Contents

Nash Equilibrium Scenarios for Russian Automotive Market Development

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Abstract This paper considers the problem of Russian car market development under interested parties' interaction. For this purpose the main players' strategies and payoffs are defined and an extensive form of the game is built. Nash equilibrium scenarios are found and interpreted in terms of individual and cooperative points of view.

Keywords: car industry, extensive form of the game, nash equilibrium, pareto equilibrium.

1. Introduction

In this paper the problem of equilibrium scenarios for Russian automotive market development and its substantial interpretation is considered. The purpose of this article is to find equilibrium scenarios for Russian automotive market development using multistage game theory model. Mathematical models of the conflicts with an account for dynamics are studied in the theory of positional games. The simplest class of positional games is a class of the finite-stage game with complete information.

In order to achieve the objective of the paper several steps are needed to be implemented. First of all, we need to conduct analysis of car industry's state of affairs in Russian Federation from interested parties' points of view. After that we will be able to identify the interested parties and players of the conflict. The third step is the revelation of the players' strategic options and building the game tree of the conflict. The final step of the paper is finding the equilibrium scenario and conclusion making.

Thus, in this paper the applicability of the game-theory modeling to the number of conflicts without access to the quantitative information is considered. The main conditions and framework of the method are stated. After all, the effectiveness of game-theory modeling is proved and the main advantages and drawbacks of the method are depicted.

2. Present Situation in Russian Automotive Industry

Nowadays the Russian automotive industry is the subject for constant discussions. On the one hand the market has grown significantly, the foreign companies have come to Russia and national companies came back to the profitability in 2010 after sharp slump in 2008-2009 (Russia Autos Report. 2011). Actually, in the end of

2009 - beginning of 2010 the market reach the historic minimum of 2006. The sales volume was 1,52 mln vehicles, including 1,355 mln passenger cars. The production also decreased noticeably from 1,79 mln vehicles in 2008 to 0,722 mln in 2009.

The passenger car market increased in 2010 by 30%, while the production grew up by 86% and reached the number of 1,108 mln cars. The difference between the market and production increase is substantial. Therefore, we can conclude, that the volume of imported cars changed insignificantly. It is also proved by the following data:

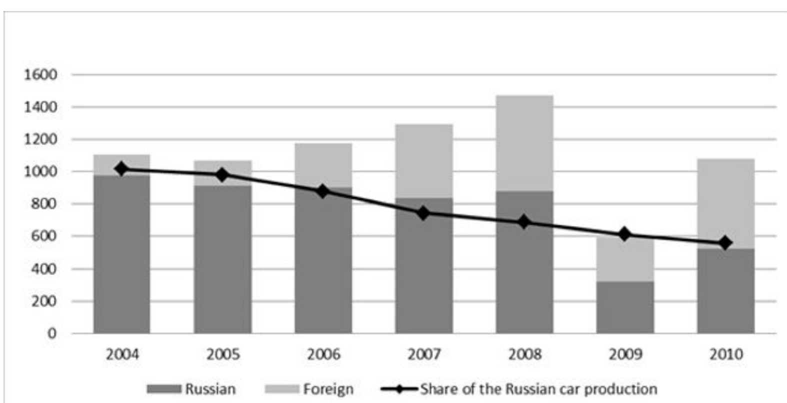
Table1: **Car sales in 2009-2010 in Russia.**

Car types	**Thousands of units**			**USD, bln**		
	2010	2009	Change (%)	2010	2009	Change (%)
Russian cars	555	380	46%	5,0	3,4	47%
Foreign cars assembled in Russia	605	350	73%	11,8	5,9	100%
Import of new foreign cars	600	625	-4%	16,9	16,5	2%
Total	1760	1355	30%	33,7	25,8	31%

The reason of Russian cars sales volume increase by 46% (to 555 thousands of units) is the Utilization program, held by Russian authorities. This program assumes the discount allotment that covers the particular car models for those customers, who utilized their old cars. Within this program 270 thousands vehicles were sold (Ministry of Industry and Trade. 2011), therefore such a program boosted the market demand. According to the Minpromtorg, approximately 80% of cars in the program were the AvtoVAZ production. In 2011 the government is planning to subsidize additionally 100 thousand cars. After that the program will be scraped gradually in order to avoid sharp decrease in sales.

Despite the fast growth in 2010, the market share and production share of the Russian car producers have been decreasing since 2003 (Figure 1).

Figure1: Car production in Russia (thousands of units)



Car production of foreign companies on the Russian Federation territory has risen during last several years In fact, the foreign car producers, who assemble car in Russia, had the most substantial growth in 2010 (116%) due to the import substitution (Rut S. 2011). In 2005 the Russian authorities impose the so called "industrial assembly" regime. According to this regime the multistage production localization of the vehicles and auto components was considered (Strategy of the Russian industry development till 2020. 2010) in order to substitute the part of the direct import by the Russian-made products. Within this regime a lot of car makers came to Russia and start to assemble the cars on the territory of Russian Federation. The terms of industrial assembly required from the foreign carmakers the production capacities of 25 thousands cars per year and the production localization at the 30% level (including car painting, welding and assembly) (Reus A. 2010.). If the foreign companies complied with these rules, they obtained the right of the components duty-free transportation in Russian Federation. The customs duties on finished products were raised to 30% to make the regime functional.

Nevertheless, according to the report, presented in "Strategy of the Russian Automotive Industry Development till 2020", this regime "didn't create the premises for development of economically reasonable up-to-date production facilities in the Russian auto components' industry". Hence, despite the increasing investments into Russian economy, this regime wasn't sufficiently effective for industry's sound development.

In fact, the present situation in the Russian automotive industry reveals the number of problems that should be solved for effective and sound car industry development. The main problems of Russian car industry are listed below:

- Lack of R&D and absence of innovations implementation experience negatively influences on quality and assortment of the products offered;
- inefficient usage of production facilities - on the one hand the production facilities are incapable to satisfy the internal demand of Russian market, but on the other hand they remain uncharged;
- underdevelopment of the Russian auto component's industry (because of the lack of competition) results in unsatisfactory quality and narrow assortment of the Russian-made components (Strategy of the Russian industry development till 2020. 2010);
- total deterioration of the factories' capital funds and technology obsolescence at the native production facilities leads to lag in technology within the industry and "low level of the Russian companies' investment appeal" (Sharovatov D. I. 2007);
- management ineffectiveness consists of huge bureaucratic system of the administrative staff and low quality of production and human resource management;
- lack of flexibility and slow adaptation of the Russian carmakers to fast-changing environment is resulted from the complex vertically-integrated organizational structures and inexperience in terms of the globalization. Consequently, for native companies it is impossible to resist the increasing competition of the foreign carmakers on Russian market (Sharovatov D. I. 2007);
- insufficient industry legislation, including absence of the clear integrated customs regulation policy hinder the industry development.

In order to solve the following problems the "Strategy of the Russian Automotive Industry Development till 2020" was elaborated by Ministry of Economic Development and Ministry of Industry and Trade in March 2010.

Based on the legislative acts and other enactments for Russian socio-economic development and on the conducted analysis of the current Russian automotive market conditions, the Strategy states the main purpose, goals and scenarios for Russian automotive industry development (Strategy of the Russian industry development till 2020. 2010). The main purpose of the Strategy is the maximization of the value added at the each car production stage from the steel and materials production to finished product assembly. Also in the Strategy the necessity to provide the variety and quality of the cars produced is highlighted (Ministry of Industry and Trade. 2011).

In order to achieve the purpose of the Strategy the following goals are posited:

– the satisfaction of the transportation industry needs;
– competitive recovery
– maximal localization of the auto components and vehicles production and the boosting competitiveness of the Russian auto components-makers;
– development of the technical regulation in the automotive industry and shortening of the technological lag between Russia and leaders in the automobile production;
– development of the production facilities in different regions of the Russian Federation, including Siberia and Far East;
– scientific base establishing in order to conduct R&D and find the opportunities for new cars and components design and construction;
– reformation of the education system for automotive industry;
– legislation improvement in the automotive industry.

In the Strategy the list of the main activities was also defined in order to reach the objectives. In includes stimulating demand activities for Russian market growth, different tariff and non-tariff measures for import diminution, stimulation of the localization level increase, establishing different joint ventures between native and foreign companies, legislative base elaboration, etc. Difference in views of interested parties can be noticed while considering these activities. According to the UK analytic company Business Monitor International import tariffs and new terms of industrial assembly won't be effective measure and can alienate investments from Russia. Moreover, modified terms of industrial assembly mismatch with the World Trade Organization requirements and can impede the entry of Russia in WTO.

New terms of industrial assembly came into effect the 4th of February 2011 (New terms of the industrial assembly. 2011). They assume that assembling facilities should be at minimum capacity of 300 thousands vehicles per year. In addition, investments in the establishing new facility should be at least $ 750 mln., while investments for modernization of an existing facility should be at least $ 500 mln. The other term of the industrial assembly regime is the gradual localization percentage increase within 4 years from 30% to 60% (Nepomnyachi A., Pismennaya E. 2010). The new terms should have been signed till the 28th of February 2011 for 8 year period. Despite the fact that these terms can be applied not only to single companies, but also to the alliances, they could push away certain investors (Russia Autos Report. 2011). For example, Fiat decided to leave the Russian automotive market, Volkswagen also had some claims.

Low quality of Russian brands' products often provokes complaints of the customers, customs duties increase and toughening the terms of industrial assembly could lead to price increase. All these factors could negatively influence the demand and customers satisfaction.

3. Interest parties, their goals and policies

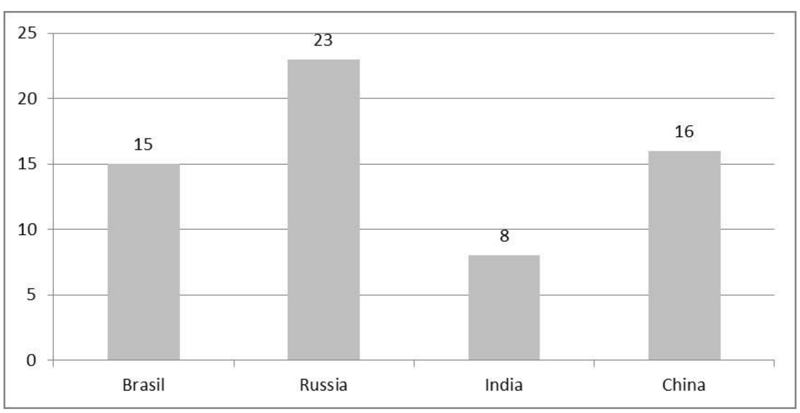
In order to analyze the conflict considered we should reveal the interested parties, players of the conflict and their goals and policies. Interested parties are the government, native carmakers, foreign carmakers, customers, producers of auto components, dealers, etc. Nevertheless, only a few interested parties have the opportunity to influence the decision making process of other parties. They are the players of the game.

The first player in the conflict is the foreign carmakers. The assumption of this paper is that the foreign carmakers represent the entity and we don't split them in smaller subsets. In the set of foreign companies we include the importers of finished products in Russia, as well as the companies with production facilities on the territory of the country.

The question of Russian car market attractiveness is disputable nowadays. According to the Roland Berger European consultancy Russian car market is very attractive for foreign carmakers in long-term perspective, in spite of its sharp decline in 2008-2009. The Government is highly interested in foreign investments attraction, and the Russian carmakers assets prices were favorable for Russian companies' acquisition. Nevertheless, according to the Mark Mobius, the well-known investor, "Russian automotive industry has a long-term potential, but its short-term perspectives are not so bright". He gives a preference to Chinese car industry in comparison with Russian automotive industry. The Business International Monitor agency in its report also mentions factors of investments and importers outflow (such as industrial assembly terms toughening).

Nevertheless, according to the Original Equipment Suppliers Association data the average price of a new car in Russian in 2008 was $ 23 000 (Roland Berger Strategy Consultants. 2010), which is the highest price in BRIC countries (Figure 2).

Figure2: **Average prices on cars in BRIC countries, 2008 (in thousands of USD).**



Attractiveness of Russian car market for foreign investors is also confirmed by CEO of "Avtostat" analytic agency Sergey Udalov. In March 2010 he claimed that "foreign companies considered the Russian automotive market as prospective, and those companies who have the possibility to invest continue to run their projects" (Buchina I. 2010). He also admitted that termination of investing into production facilities development is unprofitable for foreign companies even with significant change of the conditions. Low competitiveness of the Russian companies also benefits for foreign carmakers.

Therefore we can conclude that foreign companies presented on the Russian car market consider this market as prospective. Nevertheless, the carmakers that intent to penetrate this market and exporters are less interested in the further activity, because of the high customs duties barriers and huge expenses for newly entering companies in terms of industrial assembly regime (Nepomnyachiy A., Pismennaya E. 2010).

On the basis of the above mentioned information we can derive that foreign carmakers want to stay at the Russian car market, save their investments and share of the market. Nevertheless, to decline negative effects and additional costs because of new terms of the industrial assembling, they try to establish alliances with other carmakers. Therefore, the main direction of the foreign carmakers' policy is the temporary abstention from the large-scale activity on the Russian car market and taking the waiting attitude. Companies that didn't sign the agreement on terms of industrial assembly will wait for Russian entry in WTO and customs duties diminution (Toyota, Hyundai-Kia). The companies, which contracts came to an end, signed the new terms of the industrial assembly on the basis of compelled behavior, trying to avoid the additional costs.

The second player in the conflict is native carmakers. There we also have no subsets and consider national carmakers as a single decision-making unit. The main native carmakers are such companies as "AvtoVAZ", "GAZ Group" and Sollers. The largest Russian carmaker is AvtoVAZ with its brand "Lada" capturing 30% of the market in 2010 according to the analytic company "Avtostat" (Avtostat. 2010). The company has concluded a lot of partnership agreements with such corporations as General Motors and Renault-Nissan alliance. At the moment Renault-Nissan possess 25% shares of the Russian carmaker. Moreover, partners are planning to renew the product line of AvtoVAZ by the Renault platform (Nepomnyachiy A. 2011).

As the senior analysts of the IFC "Metropol" Andrey Rozhkov claims, GAZ Group according to its anti-crisis strategy plans to stop the production of the passenger cars and reorient its facilities for light commercial trucks production (Druzhinnin S. 2011). Nevertheless, the alliance with Volkswagen allows the company to safe presence at the passenger car market. Sollers firstly had concluded the partnership agreement with Fiat but then changed the partner for Ford.

Therefore, we can derive that in general Russian companies have no opportunities to compete with progressive foreign companies, because of the systemic problems, such as low quality of products, lack of innovations and so on. Therefore, their main goal for Russian carmakers is to avoid direct competition with foreign companies by concluding partnership agreements with them.

The third player in the conflict is the customers. They can indirectly influence on the other players decision making by accepting the products of the carmakers or refusing to buy them. It is obvious that the higher quality level and broader assortment first of all will benefit for customers of the cars. Closing the market for importers and change of the market conditions can result in the price escalation for foreign production and, consequently, can decrease customers' benefits. Therefore the main direction of customers' policy is to gain access to the foreign carmakers production at the reasonable price.

In this paper we also consider the goals and policy of the government as interested party, although we don't consider it as the player in the conflict. The Russian government establishes legislative frames and activity conditions for all other players. The authorities are interested in the Russian automotive industry development which is the significant point for the both internal and international policy.

We assume that the government has already made its decision by elaborating the Strategy of the Development till 2020 and the new terms of industrial assembly. Nevertheless, we will consider the goals and policy of the government as the important interested party and one of the main players in the further development of the conflict.

According to the Strategy the authorities have stated several possible scenarios for the Russian automotive industry development and one of them was singled out as first-priority scenario. This scenario (called "Partnership") states the following situation on the Russian car market at 2020:

– integration of the Russian carmakers in the global groups and localization of several foreign manufactures;
– goal of the industry to satisfy internal demand, and therefore low volumes of import and export;
– wide presence of the foreign producers of auto components on the Russian market within the partnerships with Russian auto components' producers;
– mutual innovation base for Russian and foreign carmakers.

Hence, the key interests of the government are the native carmakers promotion on the Russian car market, active involvement of the foreign partners, international joint ventures organization, auto components producers consolidation, etc. (Strategy of the Russian industry development till 2020. 2010). All these goals result in the main goal of the Russian government to create developed competitive automotive industry in Russia.

4. Decision Tree

In order to compose the game tree of the conflict we need to define the options of the each player which are based on its goals and policies.

For the each player the following set of alternatives is defined:

– Player A (Foreign carmakers)has three options:
  - A_1 - *actively operate on the Russian market.* In this option the foreign carmakers will develop their production facilities in Russia in order to gain the market share. This option assume the intensive collaboration between Russian and foreign carmakers.
  - A_2 - *exit from the Russian car market.* This option assumes that new entrants of the market will stop all their activities of internationalization and decrease their import. The carmakers who obtain production facilities on the Russian territory are going to get rid of the Russian assets and leave the market.

  - A_3 - *wait-and-see approach.* According to this option the foreign car makers are going to choose the wait-and-see attitude. It will be expressed in the modest investments that are inevitable for the present market share retention. Moreover, the foreign carmakers will be encouraged to make partnership agreements with Russian companies in order to gain access to the Russian production facilities and meet with the terms of the industrial assembly.
- Player B (Russian carmakers)has the two options:
  - B_1 - *collaborate with foreign carmakers.* The Russian carmakers allow foreign companies enter the market and try to create international alliances and joint ventures in order to gain access to the innovative technologies and management techniques. This collaboration can increase the Russian carmakers competitiveness and the market share.
  - B_2 - *compete with foreign carmakers.* Russian carmakers refuse to cooperate with foreign companies and try to hinder the foreign penetration on Russian car market. It can decrease the competitiveness in the industry and facilitate the activity of the native companies.
- Player C (Customers)also has two strategic options:
  - C_1 - *prefer foreign carmakers' product.* Customers can support the foreign carmakers product because of the better quality and broader assortment of the products.
  - C_2 - *prefer Russian carmakers' product.* Customers also can support the Russian carmakers if the price on the foreign products is inappropriate for them.

The decision-making process is a sequence of choices made by different players. Therefore it can be presented in the extensive-form game. Mathematical models of the conflicts with an account for dynamics are studied in the theory of positional games. The simplest class of positional games is a class of the finite-stage game with complete information.

The first step is made by the player A (Foreign carmakers). The player A chooses one of its three strategic options. Then the player B (Russian carmakers) decides which of its strategic options he is going to implement: whether he will collaborate or compete with the player A. At the last stage of the game the player C (Customers) chooses the foreign or native product.

All possible scenarios and outcomes are presented in the table 2. The number of scenarios was reduced to 7 by eliminating dead-lock scenarios. For example, it is obvious that when the player A leave the Russian market (the strategic option A2) the player B has no possibility to compete or to collaborate with the player A.

Moreover, if the player B (Russian companies) chooses the option B1 - collaborate with foreign carmakers - for the customer (the player C) there is no difference between Russian and foreign product.

On the basis of the conflict participants policies possible outcomes were ranked in terms of each player's position. For the scenarios rating the expert judgments were used. Ranking the outcomes for the foreign carmakers was based on the expert assessment of the Roland Berger Consultancy. Ranking the outcomes for the Russian carmakers and Government was based on the expert opinion of the consultants from the Avtostat analytic agency and the ministries reports. Ranking of

Table2: **Possible outcomes and scenarios of the game**

Scenario	**Outcome**
A_1B_1C	Large-scale cooperation of foreign companies with Russian carmakers within the alliances
$A_1B_2C_1$	Large-scale activity of foreign companies under competition with Russian carmakers and consumers' support of foreign products
$A_1B_2C_2$	Large-scale activity of foreign companies under competition with Russian carmakers and consumers' support of Russian products
A_2BC	Exit of foreign carmakers from Russian car market
A_3B_1C	Wait-and-see behavior of foreign carmakers under moderate cooperation with Russian carmakers
$A_3B_2C_1$	Wait-and-see behavior of foreign carmakers under competition with Russian carmakers and consumers' support of foreign products.
$A_3B_2C_2$	Wait-and-see behavior of foreign carmakers under competition with Russian carmakers and consumers' support of Russian products.

the customers' outcomes was based on the current information from the periodicals. The main criterion for the ranking was correspondence to the goals and policy of each player. Results of the ranking are presented in the Table 3.

Table3: **Ranking the scenarios**

Scenario	**Foreign carmakers (Player A)**	**Russian carmakers (Player B)**	**Customers (Player C)**	**Government**
A_1B_1C	5	7	7	7
$A_1B_2C_1$	4	1	6	2
$A_1B_2C_2$	2	3	3	6
A_2BC	1	5	1	3
A_3B_1C	7	6	5	5
$A_3B_2C_1$	6	2	4	1
$A_3B_2C_2$	3	4	2	4

Ranking was done as follows: for each player outcomes were ranked from 1 to 7, where 1 corresponded to the least preferable outcome for this player and 7 corresponded to the most preferable outcome for this player.

For foreign carmakers (Player A) the most preferable scenario is the A_3B_1C which results in the following outcome: "wait-and-see behavior of foreign carmakers under moderate cooperation with Russian carmakers". This outcome is actually in concordance with the policy of the player A and allows to avoid additional costs while attaining the current market share. Hence, the rank of the scenario A_3B_1C is 7 for the player A.

The least preferable scenario for the foreign car makers is the scenario A_2BC resulting in the outcome "exit of foreign carmakers from Russian car market", because the foreign companies have already invested in the Russian car industry huge amount of the capital which can be lost if this scenario is realized. Hence, the rank of the scenario A_2BC is 1 for the player A.

For Russian carmakers the most preferable scenario is scenario A_1B_1C resulting in the outcome "Large-scale cooperation of foreign companies with Russian carmakers within the alliances", because the Russian carmakers gain the access to the innovative and advanced technologies, R&D of foreign companies and can avoid the severe competition. The least preferable scenario for player B is $A_1B_2C_1$, which assumes the large-scale activity of foreign companies under competition with Russian carmakers and consumers' support of foreign products. The support of the customers predetermines the result of the competitive struggle in favor of foreign carmakers.

For player C (Customers) the most preferable scenario is also A_1B_1C and it results into following outcome: "Large-scale cooperation of foreign companies with Russian carmakers within the alliances". They can gain access to the broad range of the high quality products. In addition, the price will be lower than in terms of competition, because of the partnerships and common investments into production and innovations. The least preferable scenario for customers will be A_2BC. It leads to the outcome "Exit of foreign carmakers from Russian car market", when there will be only the cheap and low quality products of Russian carmakers on the market, which cannot fully satisfy the needs of the customers.

For the government the most preferable scenario is A_1B_1C, because the active collaboration of foreign and Russian carmakers enables to develop the native automotive industry by adoption of advanced technologies and enhancing the quality level of the finished products. The least preferable scenario for government is $A_3B_2C_1$ that results in "wait-and-see behavior of foreign carmakers under competition with Russian carmakers and consumers' support of foreign products". The Russian carmakers cannot compete with foreign companies because of the technological lag and other systematic problems of Russian automotive industry and therefore the native companies could be fully acquired by foreign carmakers or go bankrupt in the long-term perspective.

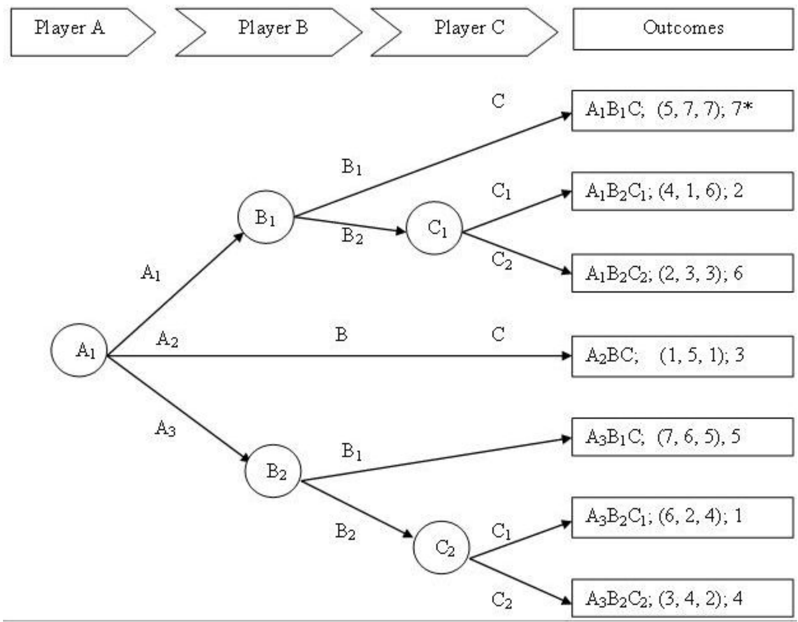
The process of players' decision-making as well as the search for equilibrium is presented in the form of decision tree - Figure 3.

5. Game-theoretic Analysis of the Model with Rankings

It is necessary to make game-theoretic analysis of the game, taking into account that all the players act rationally

Let $\nu_A(x)$,$\nu_B(x)$, $\nu_C(x)$ be the rankings (as here we have no payoffs) in the subgames $\mathit{\Gamma}_x$ in the situation of Nash equilibrium. In the subgame $\mathit{\Gamma}_{C_1}$ there is one Nash equilibrium $A_1B_2C_1$ where $\nu_A(C_1) = 4$, $\nu_B(C_1) = 1$, $\nu_C(C_1) = 6$. In the subgame $\mathit{\Gamma}_{B_1}$ there is also one Nash equilibrium A_1B_1C where $\nu_A(B_1) = 5, \nu_B(B_1) = 7, \nu_C(B_1) = 7$. In the subgame $\mathit{\Gamma}_{C_2}$ there is one Nash equilibrium $A_3B_2C_1$ where $\nu_A(C_2) = 3, \nu_B(C_2) = 4, \nu_C(C_2) = 2$. In the subgame $\mathit{\Gamma}_{B_2}$ there is one Nash equilibrium A_3B_1C where $\nu_A(B_2) = 7, \nu_B(B_2) = 6, \nu_C(B_2) = 5$. In the overall game there is a situation of absolute Nash equilibrium A_3B_1C, where $\nu_A(A_1) = \nu_A(B_2) = 7, \nu_B(A_1) = \nu_B(B_2) = 6, \nu_C(A_1) = \nu_C(B_2) = 5$. The outcome of the scenario A_3B_1C is the wait-and-see behavior of foreign carmakers under moderate cooperation with Russian carmakers. This scenario is absolute Nash equilibrium, which shows the dynamic sustainability of this equilibrium (it proves

Figure3: **Decision tree**



***ranking for the government that is not a player of the game but has some interest in the conflict and can influence the further stages of the conflict**

realizability of NE-scenario).

Now it is necessary to find Pareto optimal scenarios (which cannot be changed in the negotiations process without making the one player's payoff worse). There are two of them in this game: $A_1B_1C(5, 7, 7)$ and $A_3B_1C(7, 6, 5)$.

Therefore it is seen that there is one Pareto optimal scenario coinciding with the absolute Nash equilibrium (A_3B_1C) and the other one (A_1B_1C) does not dominate it. The Foreign companies are the main player in this conflict. In the equilibrium scenario it reaches the goal to the whole extent because the ranking of foreign companies in this outcome is 7 out of maximum 7 points. Moreover, this result is also on the second preference place for Russian carmakers.

The conducted analysis also depicts that government policy of Russian automotive industry regulation is quite effective (5 of maximum 7 points). Nevertheless, we can conclude, that the development of the car industry would be less active as the government supposed, because the rank of the equilibrium situation for government is 5 out of 7 and consequently the governmental goals are not fully achieved.

Moreover, the policy will be effective and equilibrium will be stable only with holding all conditions in the industry unchanged. The alteration in the conditions, such as legislation improvement or WTO entry, is likely to change the equilibrium scenario and put the conflict out the next stage.

6. Conclusion

Hence, the analysis of the current situation in the Russian automotive industry revealed a certain conflict between different interested parties: foreign carmakers, Russian carmakers, customers and government. In order to find equilibrium scenario in the Russian car industry conflict the game-theoretic model was built. The formulated scenarios were ranked according to different interested parties' positions. Ranking was based on the conducted analysis of the current situation and on the interests and policy's directions of players.

The main conclusions can be derived from the carried out game-theoretic analysis of the conflict. First of all, the game-theory modeling is effective method of the forecasting conflict situations. Using this method in other conflict situations can facilitate negotiations and the decision-making process.

The game-theory modeling allows us to find the equilibrium scenario with only usage of non-numeric and non-complete information. This feature of the model makes it applicable to the conflicts, where the quantitative analysis cannot be carried out.

It also has its own drawbacks. The most significant of them is the dependency on the conditions invariability. The changes in the conditions result in model changes and thus can influence the change of equilibrium scenario. Nevertheless, if we possess the information about possible changes, we can build the game-theory model for further stages of the conflict.

Therefore, described approach can be considered as the basis for the model of choosing the company's strategies in the conflict of interests of parties concerned.

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Static Models of Corruption in Hierarchical Control Systems

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Abstract Static game theoretic models of corruption in two- and three-level control systems and their applications are considered. Some concepts concerned with corruption are formalized. Several model examples are investigated analytically.

Keywords: corruption, hierarchical control systems, hierarchical games, optimization

1. Introduction

Corruption is a social-economic phenomenon that exerts a negative influence to the social processes. In the modern Russia corruption is one of the main threats to the successful social-economic transforms.

The basic pattern in game theoretic modeling of corruption is the hierarchical system principal (she) - supervisor (he) - agent (she). The pioneer work in the mathematical modeling of corruption is a paper by (Rose-Ackerman, 1975). Among many other papers we may call (Bac, 1996), (Bag, 1997), (Drugov, 2010), (Hindriks et al., 1999), (Lambert-Mogiliansky, 1996), (Mookherjee and Png, 1980), (Mishra, 2002), (Wilson and Damania, 2005). In those papers such topics as static and dynamic (multistage) corruption models, capture and extortion, grand and petty corruption, economic, political and corporative corruption, competence among bureaucrats, collusion between bureaucrat and supervisor, briber's dilemma, identification of the model parameters and many others are considered.

In this paper corruption is modeled on the base of the concept of sustainable management (Ougolnitsky, 2002, 2011); some results are presented in (Ougolnitsky, 2011). The following propositions are accepted.

1. Both principal and supervisor use methods of compulsion (administrative and legislative impacts) and impulsion (economic impacts); in the mathematical formalization compulsion restricts the set of admissible strategies meanwhile impulsion causes an effect to the payoff function of the followed player.

2. There are some values of the administrative and economic impacts which assure the conditions of homeostasis for the controlled system; the achievement of the target values is the main task of the principal in her struggle with corruption.

3. From one side, the corruption is a threat to the homeostasis because it is advantageous for the briber to weaken the requirements of homeostasis in exchange for the bribe. From the other side, corruption is a specific form of feedback in the hierarchical control systems due to which the control variables become functions of the bribe.

Unlike the majority of the papers in this domain deterministic static models of corruption mostly based on Germeyer's theory (Gorelik and Kononenko, 1982) are examined below.

2. General propositions and a model example of the administrative corruption

Let's consider a static model of the administrative corruption in the two-level hierarchical control system of the type supervisor - agent:

$$g_0(s,u,b) \to max,\ 0 \le s \le \overline{s} \le 1; \tag{1}$$

$$g_1(s,u,b) \to max,\ 0 \le u \le 1-s \le 1,\ 0 \le b \le \overline{b} \le 1;$$

$$\frac{\partial g_0}{\partial u} \ge 0,\ \frac{\partial g_0}{\partial b} \ge 0,\ \frac{\partial g_0}{\partial s} \le 0,\ \frac{\partial g_1}{\partial u} \ge 0,\ \frac{\partial g_1}{\partial b} \le 0,\ \frac{\partial g_1}{\partial s} \le 0;$$

where s is a quota (supervisor's control variable); $\overline{s}$ - the maximal admissible value of the quota; s_0 - the legal value of the quota; b - a bribe; $\overline{b}$ - the maximal admissible value of the bribe; u - an agent's action; $0 \le u \le a$ - the condition of homeostasis for the controlled system (not described explicitly in the static model). It is natural to suppose that $s_0 = 1-a$, i.e. the legal value of the quota assures the condition of homeostasis.

A function $s(b)$ describes bribery if it does not increase on the segment $[0,1]$ and $\exists b_0\ :\ s(b_0) < s_0$, i.e. for the bribe the supervisor is ready to weaken the legal requirements and therefore to create conditions for the violation of homeostasis.

We shall speak about the capture if $s(0) = s_0$ and about the extortion if $s(0) > s_0$. A case $s(0) < s_0$ (an absence of the legislative control) is not considered. In the case of capture a basic set of services is guaranteed while additional indulgences are provided for a bribe. In the case of extortion a bribe is required already for the basic set of services (in this model this set is characterized by the condition $s = s_0$). For the parametrization of bribery dependence it is often convenient to use a linear function $s(b) = A - Bb$ where a parameter $A \ge s_0$ characterizes an initial level of corruption ($A = s_0$ corresponds to the capture and $A > s_0$ to the extortion), and a parameter $B \ge 0$ characterizes a sensitivity to the bribe ($B = 0$ means that corruption is completely absent, and the sensitivity increases when B increases).

The briber's behavior is characterized by tractability and greed. The characteristic of tractability is a parameter $s_{min} = \min_{0 \le b \le 1} s(b)$, and the characteristic of greed is a parameter $b_{min}\ :\ s(b_{min}) = s_{min}$. Thus, the tractability determines a value of the maximal deviation from the legal requirements in exchange for a bribe, and the greed - a cost of the deviation. A conditional classification of the tractability and greed is given in Tables 1 and 2.

Let's consider the following problem as an example of the model (1):

$$g_1(s,u,b) = bf(u) \to max,\ 0 \le s \le \overline{s} \le 1; \tag{2}$$

$$g_2(s,u,b) = (1-b)f(u) \to max,\ 0 \le u \le 1-s,\ 0 \le b \le \overline{b} \le 1,$$

where $f(u)$ is a production function. Because the production function $f(u)$ does not decrease then the optimal agent's action is $u^* = 1-s$, and the problem of compulsion (2) reduces to the Germeyer game Γ_2 (Gorelik and Kononenko, 1982) in the form

$$g_1(s,b) = bf(1-s) \to max,\ 0 \le s \le \overline{s} \le 1;$$

Table1: Tractability levels for the capture (C) and extortion (E)

s_{min}						
	C	s_0	$(s_0/2, s_0)$	$s_0/2$	$(0, s_0/2)$	0
	E	$\overline{s}$	$(\overline{s}/2, \overline{s})$	$\overline{s}/2$	$(0, \overline{s}/2)$	0
Tractability		Minimal	Low	Middle	High	Maximal

Table2: Greed levels

b_{min}	0	$(0, \overline{b}/2)$	$\overline{b}/2$	$(\overline{b}/2, \overline{b})$	$\overline{b}$	$(\overline{b}, 1]$
Greed	Minimal	Low	Middle	High	Utmost	Superutmost

$$g_2(s,b) = (1-b)f(1-s) \to max,\ 0 \le b \le \overline{b} \le 1$$

Using the Germeyer theorem when $f(u) = \sqrt{u}$ we get:

$$s^D(b) \equiv 0;\ s^p(b) \equiv \overline{s}; L_2 = \sqrt{1-\overline{s}} \max_{0 \le b \le \overline{b}} (1-b) = \sqrt{1-\overline{s}};\ E_2 = \{0\};$$

$$D_2 = \{(s,b)\ :\ (1-b)\sqrt{1-s} > \sqrt{1-\overline{s}}\};\ K_2 = \max_{0 \le s \le \overline{s}} g_1(s,0) = 0;\ K_1 = \sup_{D_2} b\sqrt{1-s}$$

The global maximum of g_1 is achieved if $s = 0,\ b = \overline{b}$. As the players' interests coincide in s then $s^* = 0$; farther if $1 - \overline{b} > \sqrt{1-\overline{s}}$ then $b^* = \overline{b}$ else from the condition $1 - \overline{b} > \sqrt{1-\overline{s}}$ it follows $b^* = 1 - \sqrt{1-\overline{s}} - \varepsilon$.

Thus,

$$b^\varepsilon = \begin{cases} \overline{b}, & \overline{b} < 1 - \sqrt{1-\overline{s}} \\ 1 - \sqrt{1-\overline{s}} - \varepsilon, & \text{otherwise,} \end{cases}$$

i.e. in any case $b^\varepsilon = 1 - \sqrt{1-\overline{s}} - \varepsilon\ = K_1$.

As far $K_1 > 0 = K_2$ then the maximal guaranteed payoff of the supervisor is equal to K_1 , and his ε - optimal guarantying strategy has the form

$$\tilde{s}^*(b) = \begin{cases} 0, & b = 1 - \varepsilon - \sqrt{1-\overline{s}} \\ \overline{s}, & \text{otherwise} \end{cases}$$

Therefore, $s_{min} = 0$ (maximal tractability), $b_{min} = b^\varepsilon$. For example, in the case $\overline{b} = \overline{s} = 1/2$ we get

$$K_1 = 1 - \sqrt{2}/2 - \varepsilon,\ \tilde{s}^*(b) = \begin{cases} 0, & b = 1 - \varepsilon - \sqrt{2}/2 \\ 1/2, & \text{otherwise.} \end{cases}$$

Thus, for those data in exchange for a relatively small fee $b \cong 0.15 - \varepsilon$ the briber is ready to cancel completely the legal requirements of homeostasis.

3. Modeling of the administrative and economic corruption in three-level control systems

For the description of economic corruption it is expedient to take into consideration another level of hierarchy, otherwise a tax and a bribe are treated equally and the bribe directed for the tax indulgence looses its sense. Then a three-level hierarchical system "principal (federal control agency) - supervisor (corrupted official) - agent (entrepreneur - bribe-giver)" arises. In this case the players' payoff functions can schematically be presented in the following form:

$$J_P = (1-p)rf(u) \to max,\ J_S = (pr+b)f(u) \to max, \tag{3}$$
$$J_A = (1-r-b)f(u) \to max,$$

where u is an agent's action; $f(u)$ is her production function; r is a tax rate; p is a share of the official's salary in the tax receipts; b is a bribe.

Let r_0 be the legal tax rate, $r(b)$ a real value of the collected taxes with consideration of the indulgence for a bribe, $\Delta r = r_0 - r(b) > 0$. Then it is evident that the condition of advantage of the bribe for the agent is $b < \Delta r$, and the condition of its advantage for the supervisor is $b > p\Delta r$. So, the general conditions of advantage of the bribe are

$$p\Delta r < b < \Delta r \tag{4}$$

and are always true because $p < 1$ (in fact even $p << 1$). Thus, in the model (3) it is always advantageous both to give and to take a bribe, and its specific value in the range (4) can be a subject of bargaining between supervisor and agent. Another variant of description of the economic corruption is also possible: to save two levels of the hierarchy supervisor - agent but to establish that the supervisor has two criteria of optimality - tax collection for the state and personal interest (bribe), for example

$$J_1 = rf(u) \to max,\ J_2 = bf(u) \to max. \tag{5}$$

If to solve this problem by maximization of the convolution

$$J = k_1 J_1 + k_2 J_2 = (k_1 r + k_2 b)f(u),$$

then in the specific case $k_1 = p,\ k_2 = 1$ (absolute priority of the bribe) the problem (5) is reduced to the agent's criterion from (3).

It is also possible to get the conditions of advantage of the administrative bribe for the two-level model in the form

$$J_S = (pr+b)f(u) \to max,\ 0 \le s \le 1 \tag{6}$$
$$J_A = (1-r-b)f(u) \to max,\ 0 \le u \le 1-s;\ 0 \le b \le 1.$$

As far $f(u)$ does not decrease then $u^* = 1-s$. Denote $f_0 = f(1-s_0),\ f_b = f(1-s(b))$, where s_0 is a legal value of quota, $s(b)$ the real value of quota with consideration of indulgence for a bribe, $\Delta f = f_b - f_0 > 0$. Then $(1-r)f_0$ is an agent's payoff without any bribe, $(1-r-b)f_b$ is her payoff in the case of bribe, so the condition of advantage of the bribe for the agent is $(1-r-b)f_b > (1-r)f_0$, or $(1-r)\Delta f > bf_b$.

The agent's payoff without a bribe is equal to prf_0 , in the case of bribe $(pr+b)f_b$, i.e. it is advantageous for the agent to give a bribe in any case. Thus, the condition of advantage of the administrative bribe in the model (6) is

$$(1-r)\Delta f > bf_b$$

In more general form a game theoretic model of control in a three-level hierarchical system can be written as

$$G(p,q_r,q_s,r,s,u,b_r,b_s) \to max,\ 0 \le p \le 1;\quad 0 \le q_r \le 1;\quad 0 \le q_s \le 1; \tag{7}$$

$$G_0(p,q_r,q_s,r,s,u,b_r,b_s) \to max,\ 0 \le q_r \le r \le \overline{r} \le 1;\quad 0 \le q_s \le s \le \overline{s} \le 1;$$

$$g(p,q_r,q_s,r,s,u,b_r,b_s) \to max,\ 0 \le u \le 1-s;\ b_r \ge 0;\ b_s \ge 0; b_r + b_s \le 1.$$

Here p is the principal's economic control variable; q_r, q_s are her administrative control variables directed to the regulation of the supervisor's economic and administrative activity respectively; r is the supervisor's economic control variable (tax); s - his administrative control variable (quota); u - the agent's action; b_r, b_s are her tax and quota bribes respectively.

It is supposed that the values of tax r_0 and quota s_0 exist which assure the conditions of homeostasis for the controlled system (not described explicitly in the static model). Functions $r = r(b_r),\ s = s(b_s)$ describe the economic and administrative corruption respectively.

The principal's task of struggle with corruption in the model (7) is solved by means of the following algorithm of two-stage optimization:
1) to fix the values of principal's control variables as parameters and to find a solution of the parametric game Γ_2 supervisor - agent;
2) to choose the values of principal's control variables which provide in the following solution of the parametric game the choice of homeostatic strategies by the supervisor r_0, s_0.

Let's consider as an example the following model in which the economic methods are not used (p-const) and denote $q_s = q,\ b_s = b$:

$$G(q,s,u,b) = -M|s - s_0| \to max,\ 0 \le q \le 1;$$

$$G_0(q,s,u,b) = bf(u) \to max,\ 0 \le q \le s \le \overline{s} \le 1;$$

$$g(q,s,u,b) = (1-b)f(u) \to max,\ 0 \le u \le 1-s;\ 0 \le b \le \overline{b} \le 1.$$

The principal's payoff function reflects an obligatory homeostatic requirement $s = s_0$, a violation of which entails an arbitrary big penalty of the principal ($M >> 1$). Let's take $f(u) = \sqrt{u}$ and fix q; then a Γ_2 game of the following type arises

$$G_0(s,b) = b\sqrt{1-s} \to max,\ q \le s \le \overline{s};$$

$$g(s,b) = (1-b)\sqrt{1-s} \to max,\ 0 \le b \le \overline{b}.$$

Using the Germeyer theorem as in the case of two-level model we get a maximal guaranteed payoff of the agent $K_1 = b^{\varepsilon}\sqrt{1-q} = (1 - \sqrt{(1-\overline{s})/(1-q)} - \varepsilon)\sqrt{1-q}$ and her ε -optimal guarantying strategy

$$\tilde{s}^*(b) = \begin{cases} q, & b = b^{\varepsilon}, \\ \overline{s}, & \text{otherwise} \end{cases}$$

Thus, in this case the principal by the choice of the value $q = s_0$ can assure the condition $s = s_0$.

4. Optimization models of corruption

If the bribery function is known then corruption can be described by an optimization model. The optimization model of economic corruption has the form

$$g(b) = b + r(b) \to min,\ 0 \le b \le 1. \tag{8}$$

where b is a bribe, $r(b)$ is a function of the economic corruption (for example, a real diminution of the tax rate, i.e. absence of sanctions for the tax evasion).

So, the function $g(b)$ is treated as total cost for the tax payment and the bribe which is to be minimized by the agent.

In the case of linear parametrization $r(b) = r_0 - Ab$ the model (8) takes the form

$$g(b) = r_0 + (1 - A)b \to min,\ 0 \le b \le 1. \tag{9}$$

Here r_0 is a legal tax rate, A - the model parameter. As the function of economic corruption $r(b) = r_0 - Ab$ decreases monotonically when $0 \le b \le 1$ then $A > 0$. On the other case, the total cost $g(b)$ is not negative, therefore $A \le 1 + r_0$. Thus, $0 < A \le 1 + r_0$.

The parameter A determines qualitative characteristics of the bribe-taker behavior. If $A = 0$ then the corruption is absent completely. When A increases, the bribe-taker's tractability also increases and his greed diminishes. The threshold value is $A = r_0$: in this case $r(1) = 0$, i.e. an utmost greed provides a maximal tractability. When $A < r_0$ the greed is super-utmost, and the tractability doesn't reach its maximal value (i.e. any bribe does not deliver from a positive tax). If $A > r_0$ then the agent can avoid tax payment at all in exchange of relatively small bribe (maximal tractability and small greed).

Let's return to the solution of the problem (9). As far $dg(b)/db = 1 - A$ then when $0 < A < 1$ the function g increases monotonically and its maximal value is reached on the left bound of the admissible range of its values: $g_{min} = g(0) = r_0$. Respectively, if $1 < A < 1 + r_0$ then the function g decreases monotonically and its maximal value is reached on the right bound of the admissible range: $g_{min} = g(1) = 1 + r_0 - A < r_0$. In the degenerate case $A = 1$ it is true $g(b) \equiv r_0$(any bribe is useless, the corruption is absent).

So, in this case the parameter A also plays a key role and determines two qualitatively different strategies of the agent's behavior. If $0 < A < 1$ then total agent's cost $g(b)$ increases, it is rational to not give a bribe and honestly pay taxes in the rate r_0. If $1 < A < 1 + r_0$ then $g(b)$ diminishes and it is economically expedient to pay a bribe and to have a total payment in the sum $1 + r_0 - A < r_0$.

In the case of quadratic parametrization of the function of economic corruption $r(b) = r_0 - Ab^2 (0 < A \le 1 + r_0)$ a qualitative situation is not changed. As in the linear case we have

$$g_{min} = \begin{cases} g(0) = r_0, & 0 < A < 1, \\ g(1) = r_0 + 1 - A, & 1 < A < 1 + r_0 \end{cases} \tag{10}$$

So, when $0 < A < 1$ there is no reason to propose a bribe, and the total cost r_0 is minimal if tax is payed; when $1 < A < 1 + r_0$ it is rational to give the maximal bribe $b = 1$, and the total cost is equal to $1 + r_0 - A < r_0$.

Let's now consider a power parametrization of the function of economic corruption in the form $r(b) = r_0 - A\sqrt{b}$. Then a problem of minimization of agent's total cost is

$$g(b) = r_0 + b - A\sqrt{b} \to min,\ 0 \le b \le 1.$$

In this case

$$g(0) = r_0,\ g(1) = r_0 + 1 - A,\ \frac{dg(b)}{db} = 1 - \frac{A}{2\sqrt{b}},\ \frac{d^2g(b)}{db^2} = \frac{A}{4b^{2/3}},$$

therefore $b^* = A^2/4$ is a minimum point. Notice that $g(b^*) < g(1)$, so the minimal value of the cost function is equal to $g_{min} = g(b^*) = r_0 - A^2/4$. To provide non-negativity of the cost it is necessary to require that $A \le 2\sqrt{r_0}$. So, it is always advantageous for the agent to give the bribe $b^* = A^2/4$ that reduces the total cost to the value $g_{min} = g(b^*) = r_0 - A^2/4 < r_0$ (in the limit case $A = 2\sqrt{r_0}$ to zero).

Thus, when a function of the economic corruption $r(b) = r_0 - Ab^k$ is given then the results of model analysis depend both on values of the parameter A and on values of the parameter k. When $k = 1, 2$ a minimal value of the total cost is determined by the expression (10), i.e. if $0 < A < 1$ then there is no reason to propose a bribe, an honest tax payment leads to the minimal cost value r_0 ; if $1 < A < 1 + r_0$ then it is expedient to give the maximal bribe $b = 1$, and then the total cost is equal to $1 + r_0 - A < r_0$. If $k = 1/2$ then it is always advantageous to the agent to give the bribe $b^* = A^2/4$ and reduce the total cost to the value $g_{min} = g(b^*) = r_0 - A^2/4 < r_0$. It can be supposed that the results remain valid for any $k \ge 1$ and $k < 1$ respectively.

An optimization model of the administrative corruption has the form

$$g(b) = (1 - b)f(s(b)) \to min,\ 0 \le b \le 1. \tag{11}$$

where b is a bribe, $s(b)$ is a function of administrative corruption, f is a production function of the agent - bribe-giver. As far the production function increases, its argument is equal to the value of right border of the admissible range of the agent's strategies set restricted by the value of corrupted quota $s(b)$.

In the case of linear parametrization of the function of administrative corruption $s(b) = s_0 + Ab$ and linear production function $f(x) = x$ the model (11) takes the form

$$g(b) = (1 - b)(s_0 + Ab) \to max,\ 0 \le b \le 1. \tag{12}$$

As in the case of economic corruption the parameter A determines qualitative characteristics of agent's behavior. If $A = 0$ then the corruption is completely absent. When A increases the tractability also increases and the greed diminishes. The threshold value is $A = 1 - s_0$: in this case $s(1) = 1$, i.e. an utmost greed provides a maximal tractability. If $A < 1 - s_0$ then the greed is super-utmost and the tractability does not reach its maximal value (some quota is obligatory for any bribe). If $A > 1 - s_0$ then the agent can ignore a quota in exchange for relatively small bribe (maximal tractability and small greed).

Let's return to the solution of the problem (12). We get

$$g(0) = s_0,\ g(1) = 0,\ \frac{dg(b)}{db} = A - s_0 - 2Ab,\ \frac{d^2g(b)}{db^2} = -2A < 0,$$

therefore $b^* = (A - s_0)/(2A)$ is a maximum point, $g(b^*) = (A + s_0)^2/(4A) \ge g(0)$.

Notice that

$$b^* \begin{cases} > 0, & A > s_0, \\ < 0, & A < s_0, \end{cases} \quad \text{so} \quad g_{max} = \begin{cases} g(b^*), & A > s_0 \\ g(0), & A < s_0. \end{cases}$$

Thus, in this case the parameter A also plays a key role and determines two qualitatively different strategies of the agent's behavior. If $A < s_0$ then there is no reason to propose a bribe, the agent's income reaches its maximal value s_0 when $b = 0$. However, if $A > s_0$ then the optimal bribe value is $b^* = (A - s_0)/(2A)$, and the maximal income is equal to $(A + s_0)^2/(4A) \geq s_0$.

In the case of power parametrization of the production function $f(x) = \sqrt{x}$ and linear function of the administrative corruption $s(b) = s_0 + Ab$ the qualitative situation is not changed, namely

$$g_{max} = \begin{cases} g(b^*), & A > 2s_0, \\ g(0), & A < 2s_0. \end{cases}$$

If $A < 2s_0$ then there is no reason to propose a bribe, the agent's income reaches its maximal value $\sqrt{s_0}$ when $b = 0$. If $A > 2s_0$ then the optimal bribe value is $b^* = (A - 2s_0)(3A)$, and the maximal income is equal to $2\sqrt{(A + s_0)^3}/(3A\sqrt{3}) \geq \sqrt{s_0}$.

An inductive hypothesis arises that for any parametrization $g(b) = (1 - b)(s_0 + Ab)^k,\ k \leq 1$ the maximal income is determined by the expression

$$g_{max} = \begin{cases} g(0), & A < s_0/k, \\ g(b^*), & A > s_0/k \end{cases}, \qquad b^* = \frac{kA - s_0}{(1 + k)A}$$

A prove of the hypothesis and the investigation of other classes of the functions of administrative corruption is a subject of further research.

5. Modeling of corruption in water resource quality control systems

As an extended example we model corruption in the three-level water resource quality control system which includes the following control elements: federal control center (principal), regional or local control agencies (supervisor), industrial enterprises (agents), as well as controlled water system (river).

The agents tend to maximize their profit from production and throw pollutants to the river. The supervisor determines a fee for pollution and tends to maximize the penalties collected from the agents. The principal should assure a homeostasis of the river. The interests of principal and supervisor are different, and the supervisor may be interested in bribes from agents. In exchange for bribes the supervisor reduces the fee for pollution. The principal should provide such conditions that it will be economically advantageous for the supervisor to provide the homeostatic requirements even with corruption.

The principal can charge penalties on the supervisor and the agents for corruption. The value of penalty depends on scale factors determined by the principal. If the scale factors are big, i.e. a probability of bribe detection and a power of the punishment are big then the economic reason of corruption disappears. In the same time, when the scale factors increase, the cost of principal's control also increases. The considered control method is impulsion (Ougolnitsky, 2002, 2011).

Let's consider the case of one pollutant and one agent. It is supposed that the river is in homeostasis if some standards of water quality for the river

$$0 \leq B \leq B_{max} \tag{13}$$

and the sewage water

$$\frac{W(1-P)}{Q^0} \le Q_{max} \tag{14}$$

are satisfied, where B is a concentration of the pollutant in the river water; Q^0 is a water flow for the agent; W is an amount of pollutant in sewage before its refinement; P is a share of the pollutant removed from the sewage due to its refinement; values of B_{max}, Q_{max} are given.

Suppose that a concentration of the pollutant in the river is calculated by the formula

$$B = B_0 e^{-k} + W(1-P) \tag{15}$$

where $B_0, k = const.$

Besides assuring the homeostatic conditions, the principal tends to maximize her payoff function

$$J_P = (1-P)W\{F(T^0)H - h(L) + L\delta(\alpha_1(b) + \alpha_2(b))\} \to \max_{H,L} \tag{16}$$

Here $F(T^0)$ is a function of payment per unit of pollution when corruption is present;

$$T^0 = \begin{cases} T + S - \delta a(b), & if\ T + S - \delta a(b) \ge 0, \\ 0 & \text{otherwise;} \end{cases}$$

T is a payment per unit of pollution when corruption is absent; δ is equal to one if a bribe is accepted, and to zero, otherwise; b is a bribe; $a(b)$ is a function of the bribe efficiency; $S = 0$ in the case of capture and $S > 0$ in the case of extortion; H is a share of agent's payments for pollution belonging to the principal; $\alpha_1(b), \alpha_2(b)$ are penalty functions for corruption for supervisor and agent respectively; L is a scale factor which permits to vary the punishment for corruption; $h(L)$ is principal's function of expenditures for determining the scale factor per unit of pollution.

The supervisor payoff function has the form

$$J_S = (1-P)W\{F(T^0)(1-H) - L\delta\alpha_2(b) + \delta b\} \to \max_{T,\delta} \tag{17}$$

The supervisor chooses the values of pollution fees and decides whether it is advantageous for him to take bribes proposed by the agent.

The agent tends to maximize her profit in the presence of corruption, i.e.

$$J_A = zR(\Phi) - (1-P)W\{F(T^0) + L\delta\alpha_1(b) + b\delta\} - WC_A(P) \to \max_{P,b} \tag{18}$$

where $C_A(P)$ is agent's function of expenditures for refinement per unit of pollution; Φ is her production resource; $R(\Phi)$ is agent's production function; z is agent's profit per unit of the product.

Assume that an amount of pollution is a linear function of agent's product with a constant coefficient β, i.e.

$$W = \beta R(\Phi) \tag{19}$$

where

$$R(\Phi) = \gamma\Phi^\eta;\ \eta, \gamma = const; 0 < \eta < 1 \tag{20}$$

Optimization problems (18) - (20) are solved with the following restrictions:

$$0 \le P \le 1 - \varepsilon; \quad 0 \le b \le b_{max}; \tag{21}$$

$$0 \le H \le 1-\theta; \quad 0 \le L \le L_{max}; \tag{22}$$

$$\delta = \begin{cases} 0, \\ 1; \end{cases} \quad 0 \le T \le T_{max}; \tag{23}$$

where $T_{max}, b_{max}, L_{max}$ are given values; $\varepsilon > 0$ is a constant characterizing technological capacity of sewage refinement; $\theta < 1$ is a minimal share of payments collected from the agent which the supervisor gives to the principal.

It is follows from (13) - (15) that to provide the homeostasis it is sufficient to satisfy the inequality

$$p_{sd} \le P \le 1-\varepsilon; \tag{24}$$

where

$$p_{sd} = max(1 - \frac{Q_{max}Q^0}{W}; 1 - \frac{B_{max} - B_0 e^{-k}}{W}).$$

In the model (16) - (24) a method of impulsion is used by both the principal and the supervisor; capture ($S = 0$) and extortion ($S > 0$) are described. The model is analyzed by Lagrange multipliers method after transition from restrictions in the form of inequalities to the restrictions in the form of equalities. Some model results are presented in the Appendix.

6. Conclusion

The models of corruption in hierarchical control systems based on original concept of sustainable management are considered. The structural pattern of modeling is a principal - supervisor - agent construct. The corruption may be described from the positions of bribe-giver (agent), bribe-taker (supervisor), and bribe-fighter (principal). In the first case a bribery function is supposed to be known and optimization models arise. In the second case a hierarchical game of the type Γ_2 is considered. In the third case the principal seeks for the values of her control parameters which assure the requirements of homeostasis for the found optimal strategy of the supervisor. The supervisor's characteristics are his tractability (a value of the possible indulgence for the violation of legal requirements in exchange of a bribe) and greed (a price of the indulgence). Capture and extortion as two kinds of bribery are differentiated. In the case of capture a basic set of services is guaranteed while additional indulgences are provided for a bribe. In the case of extortion a bribe is required already for the basic set of services. The conditions of advantage of bribe-taking and bribe-giving are described.

It is shown that for different values of structural and numerical parameters of the optimization and game theoretic models qualitatively different strategies of the agent and the supervisor arise. For some values of the parameters it is more rational to act honestly (to pay taxes and to obey quotas), meanwhile for other values a bribe permits to reduce costs or increase income. Therefore, a development of methods of the model identification plays an important role in the struggle with corruption.

An investigation of the dynamic models of corruption in hierarchical control systems is under development.

Appendix

Let's consider some examples of investigation of the model (16) - (24) for the following input functions:

$$F(T) = T;\ a(b) = A_1 b;\ \alpha_1(x) = \vartheta_1 x;\ \alpha_2(x) = \vartheta_2 x;\ h(L) = kL;$$

$$C_A(P) = D\frac{P}{1-P};\ A_1, \vartheta_1,\ \vartheta_2,\ k,\ D = const$$

Denote the optimal strategies of different control elements by $\delta^*, T^*, b^*, P^*, L^*, H^*$.

If $A_1 - 1 - \vartheta_1 L^* > 0$ then from the point of view of the agent a bribe is effective and its value is maximal: $b^* = b_{max}$, otherwise a bribe is not proposed: $b^* = 0$. For the supervisor for any input data $T^* = T_{max}$.

Denote

$$p^0(b, L) = 1 - \sqrt{\frac{D}{T_{max} + S - b(A_1 - 1 - \vartheta_1 L)}}$$

Then

$$P^*(b, L) = \begin{cases} 0, & p^0(b, L) < 0, \\ p^0(b, L), & 0 \le p^0(b, L) \le 1 - \varepsilon, \\ 1 - \varepsilon, & p^0(b, L) > 1 - \varepsilon \end{cases}$$

It is seen from the formula that corruption reduces the power of refinement of sewage water optimal for agent.

For simplicity consider the case

$$\theta A_1 < 1,\ \varepsilon^2(T_{max} + S - b_{max}(A_1 - 1 - \vartheta_1 L_{max})) < D < T_{max} + S - b_{max}(A_1 - 1)$$

Then it is advantageous for the supervisor to accept bribe in the absence of control $L = 0$ and $H = 1 - \theta$ and, besides, $P^* = p^0(b, L)$.

Example 1. Assume that corruption is not punished, i.e. $k = \vartheta_1 = \vartheta_2 = 0$. If $H^* > 1 - 1/A_1$ then the supervisor accept a bribe and his payoff in the presence of corruption is greater than in its absence ($\delta^* = 1$), otherwise the bribe is rejected ($\delta^* = 0$).

Thus, even without an administrative control the principal can create such economic conditions that it will be not rational for the supervisor to take bribes from the agent.

If $p^0(b_{max}, 0) \ge p_+;\ A_1 > 1$ and

$$(1-\theta)(T_{max} + S - b_{max}A_1)\sqrt{\frac{D}{T_{max} + S - b_{max}(A_1 - 1)}} \ge$$

$$\sqrt{\frac{D}{T_{max} + S}}(T_{max} + S)(1 - \frac{1}{A_1})$$

then corruption is profitable for the principal and

$$H^* = 1 - \theta;\ T^* = T_{max};\ \delta^* = 1;\ b^* = b_{max};\ P^* = p^0(b_{max}, 0).$$

If $p^0(b_{max}, 0) \ge p_+;\ A_1 > 1$, but

$$(1-\theta)(T_{max} + S - b_{max}A_1)\sqrt{\frac{D}{T_{max} + S - b_{max}(A_1 - 1)}} <$$

$$\sqrt{\frac{D}{T_{max} + S}}(T_{max} + S)(1 - \frac{1}{A_1})$$

then the principal is not interested in corruption. In this case she gives a part of the financial resources collected from the agent to the supervisor for whom it becomes not rational to take bribes. Then

$$H^* = \frac{1}{A_1};\ T^* = T_{max};\ \delta^* = 0;\ b^* = 0;\ P^* = p^0(0,0).$$

The same situation is observed in the case when $A_1 > 1$ and $p^0(b_{max}, 0) < p_+ \leq p^0(0,0)$. The homeostatic requirements are satisfied only without corruption therefore the principal fights again it. Optimal strategies of the players are the same as in the previous case.

If $A_1 < 1$ and $p_+ \leq p^0(0,0)$ the corruption is absent, the homeostatic requirements are satisfied and

$$H^* = 1 - \theta;\ T^* = T_{max};\ \delta^* = 0;\ b^* = 0;\ P^* = p^0(0,0).$$

Example 2. Assume that the supervisor and the agent can be punished for bribes, i.e. $k > 0;\ \vartheta_1 > 0;\ \vartheta_2 > 0$ and $A_1 > 1$.

If $(\vartheta_1 + \vartheta_2)b_{max} < k$ then $L^* = 0$. A control over the supervisor and the agent is not rational for the principal and the case is reduced to the case of the absence of punishment for bribes. Otherwise, corruption is profitable for the principal. The principal's resource assigned to the control over the supervisor and the agent can be returned only by penalties charged on them.

If $p^0(b_{max}, L_{max}) < p_+ \leq p^0(0,0)$ then homeostasis is possible only without corruption. If $1 - A_1 - \vartheta_2 L_{max} \geq 0$ then it is impossible to eliminate corruption, the method of impulsion does not work and the homeostasis is violated.

If $1 - A_1 - \vartheta_2 L_{max} < 0$ and $p^0(b_{max}, L_{max}) < p_+ \leq p^0(0,0)$ then corruption is absent and

$$L^* = L_{max};\ H^* = max(1-\theta,\ 1 - \frac{1 - \vartheta_2 L_{max}}{A_1});\ T^* = T_{max};$$

$$\delta^* = 0;\ b^* = 0;\ P^* = p^0(0,0).$$

If $(\vartheta_1 + \vartheta_2)b_{max} > k$ (the control is profitable for the principal) and $p^0(b_{max}, L_1) > p_+$, where $0 \leq L_1 \leq L_{max}$ (homeostasis is possible even with corruption) then

$$L^* = max(0,\ min(\frac{A_1 - 1}{\vartheta_1};\ \frac{1 - \theta A_1}{\vartheta_2};\ \frac{(A_1 - 1)b_{max} + D/(1-p_+)^2 - T_{max} - S}{\vartheta_1 b_{max}}));$$

$$H^* = 1 - \theta;\ T^* = T_{max};\ \delta^* = 1;\ b^* = b_{max};\ P^* = p^0(b_{max}, L^*).$$

In this case for the principal it is rational to choose the maximal value of control over the supervisor and the agent for which the system is in the homeostasis but for the supervisor and the agent is still profitable to take and give bribes.

The condition $L^* \leq (A_1 - 1)/\vartheta_1$ makes rational for the agent to give bribes, the condition $L^* \leq (1 - \theta A_1)/\vartheta_2$ - to accept it, and the condition

$$L^* \leq \frac{(A_1 - 1)b_{max} + D/(1-p_+)^2 - T_{max} - S}{\vartheta_1 b_{max}}$$

provides the homeostasis of the system.

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Pontryagin's Alternating Integral for Differential Inclusions with Counteraction

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Abstract The notion of Pontryagin's alternating integral is generalized for differential inclusions with counteraction and simplified schemes of construction of the alternating integral are proposed.

Keywords: pursuit problem, Pontryagin's method, differential inclusion, alternating sum, alternating integral, multivalued mapping, convex compact set, Nikolskiy's cup.

1. Introduction

In (Pontryagin, 1967) the new phenomena of an alternating integral was introduced. Pontryagin's second direct method for linear differential games of pursuit (Pontryagin, 1980) being based on this conception has played the great role in development of the thepry of differential games ((Azamov, 1988)-(Kurzhanskiy and Melnikov, 2000)).

In the present paper it will be studied the notion of the alternating integral for pursuit games, being described by differential inclusions $\dot{z}(t) \in -F(t, \upsilon)$, where F is a continuous multivalued mapping (Azamov, 1988). The typical example of such type of systems is a quasilinear differential game (Mishchenko and Satimov (1974)) $\dot{x} = Cx - f(u, \upsilon), u \in P, \upsilon \in Q$, which easily can be transformed to differential inclusion $\dot{z}(t) \in -e^{-tC} f(P, \upsilon)$.

Further we shall use the following notations: $I = [\alpha, \beta]$ is the fixed closed interval of time; Δ is a subsegment of I; $|\Delta|$ is the length of Δ; $cl(\mathbb{R}^d)$ ($Ccl(\mathbb{R}^d)$, respectively) is the collection of all nonempty closed (convex closed) subsets of $\mathbb{R}^d$; $cm(\mathbb{R}^d)$ ($Ccm(\mathbb{R}^d)$, respectively) is the collection of all nonempty compact (convex compact) subsets of $\mathbb{R}^d$; $H = \{z \in \mathbb{R}^d \mid \ |z| \leq 1\}$ is the unit closed ball in $\mathbb{R}^d$. $\omega = \{\tau_0, \tau_1, \tau_2, ..., \tau_n\}$ is partition of I (i.e. $\alpha = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n = \beta$, n can depend on ω); Ω is the collection of all partition of the segment I; $\Delta_i = [\tau_{i-1}, \tau_i]$; $\delta_i = |\Delta_i|$; $|\omega| = max|\delta_i|$ is the diameter of the partition ω; $\int_{\Delta_i}$ wil be shortened as $\int_i$. If X is a subset of Euclidean space, then $X[\Delta]$ denotes the collection of all measurable functions $a(\cdot) : \Delta \to X$. In the case of $\Delta = [\alpha, \beta]$, we will simply write $X[\alpha, \beta]$.

We consider the controlled differential inclusion

$$\dot{z} \in -F(t, \upsilon), \tag{1}$$

where $z \in \mathbb{R}^d, \upsilon \in Q,\ t \in I$, $Q \in cm(\mathbb{R}^q)$ and $F : I \times Q \to Ccm(\mathbb{R}^d)$ is continuous mapping. There is also given subset M, $M \subset \mathbb{R}^d$ (1) called terminal set of the system (1).

For any partition ω, $\omega \in \Omega$, we define the alternating sum $S(\omega)$, by the following recurrent scheme

$$S^0 = M,\ S^i = \bigcap_{\upsilon(\cdot)\in Q(\Delta_i)} \left[S^{i-1} + \int_i F(t,\upsilon(t))dt \right],\quad S(\omega) = S^n. \tag{2}$$

The set

$$W_\alpha^\beta(M) = \bigcap_{\omega\in\ \Omega} S(\omega)$$

is known Pontryagin's alternating integral (it was introduced in (Pontryagin, 1967), more exact definition in (Pontryagin, 1980), generalization for quasilinear games was considered (Mishchenko and Satimov (1974)), see also (Azamov, 1982)– (Kurzhanskiy and Melnikov, 2000)).

Further, when necessary, we shall indicate in notations dependence of sums and integrals not only of ω or α, β, but also of other initial data, for example $S^1(M)$, $S(\omega, P, Q)$, $W_\alpha^\beta(M, F)$. In the case $I = [0, \tau]$ we will write $W^\tau(M)$ or even W^τ. The aim of the paper is to give simplified schemes in comparison with (2).

2. Preliminary properties

Lemma 1. *(Azamov, 1982). Let a sequence $X_k \in cl(\mathbb{R}^d)$ decreases monotonically by inclusion, and $Y \in cm(\mathbb{R}^d)$. Then the equality*

$$\left(\bigcap_{k=1}^{\infty} X_k\right) + Y = \bigcap_{k=1}^{\infty} (X_k + Y)$$

is valid.

It should be noted, for any family $X_\alpha \subset \mathbb{R}^d$ and a set $Y \subset \mathbb{R}^d$ the following relation

$$(\bigcap_\alpha X_\alpha) + Y \subset \bigcap_\alpha (X_\alpha + Y). \tag{3a}$$

holds.

Lemma 2. *(Gusyatnikov, 1972). Let $M \in cl(\mathbb{R}^d)$ and a sequence of partitions $\omega_n \in \Omega$ decreases monotonically by inclusion, .. $\omega_n \subset \omega_{n+1}$, $\mid \omega_n \mid \to 0$ for $n \to \infty$. Then*

$$W_\alpha^\beta(M) = \bigcap_{k\geq 1} S(\omega_k).$$

In (Gusyatnikov, 1972) this important lemma was proved using Zorn's lemma (see also (Pshenichniy and Sagaydak, 1970)–(Polovinkin, 1979)). There we are going to give its direct proof.

Let $\omega \in \Omega$ be any partition. Values relating to partition ω_k, we indicate by index k, for example, n_k is a number of parts, $\tau_j^{(k)}$ are division point of this partition, $j = \overline{1, n_k}$. It is obvious, there is a such N, that $|\omega_k| < \frac{1}{4}\min_{1\leq i\leq n}\delta_i$ if $k > N$. Further we consider this condition is satisfied.

For each i by $j(i)$ we denote the minimum value of the index j, such that $min_{1\leq j\leq n_k} \mid \tau_j^k - \tau_i \mid$ is reached. For $k > N$, numbers $\bar{\tau}_i^{(k)} = \tau_{j(i)}^{(k)}$ will be pairwise different and form the partition $\bar{\omega}_k$, which has the same number of division points as ω. we will mark out the objects of partition $\bar{\omega}_k$ by same way of symbolization as $\bar{\tau}_i^{(k)}$. Notice that the value $\chi_k = \max_{1\le i\le n} \mid \tau_i - \bar{\tau}_i^{(k)} \mid$ characterize the deviation of $\bar{\omega}_k$ from partition ω.

It is easy to see

$$\int_{\bar{\Delta}_i} F(t, \upsilon(t))dt \subset \int_{\Delta_i} F(t, \upsilon(t))dt + 2\lambda\chi_k H, \tag{4}$$

where $\lambda = \max\{h(0, F(t, \upsilon)) \mid t \in I, \upsilon \in Q\}$.

Now it is possible to estimate partial sums $\bar{S}^i$ according to partition $\bar{\omega}_k$ through the partial sums S^i of the partition ω. By virtue of (4), we obtain

$$\bar{S}^0 = M, \bar{S}^1 = \bigcap_{\upsilon(\cdot)\in Q(\bar{\Delta}_1)} \left[M + \int_{\bar{\Delta}_1} F(t, \upsilon(t))dt\right] \subset$$

$$\subset \bigcap_{\upsilon(\cdot)\in Q(\Delta_1)} \left[M + \int_{\Delta_1} F(t, \upsilon(t))dt + 2\lambda\chi_k H\right] = S^1(M + 2\lambda\chi_k H, \omega).$$

Repeating this reasoning gives $\bar{S}(M, \bar{\omega_k}) \subset S(M + 2\lambda\chi_k nH, \omega)$. Therefore

$$\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset \bigcap_{k\ge N} S(M + 2\lambda n\chi_k H, \omega).$$

Using lemma 1 we bring the operation of intersection inwards:

$$\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset S(\bigcap_{k\ge N} (M + 2\lambda n\chi_k H, \omega).$$

Since the set M is convex closed and the number of division points n of the partition ω is not depend of k and $\chi_k \to 0$ if $k \to \infty$, then we get the inclusion $\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset S(M, \omega)$. Hence, $\bigcap_k \bar{S}(M, \bar{\omega}_k) \subset W_\alpha^\beta(M)$. The reverse inclusion is evident. Lemma 2 is proved.

Corollary 1. *Let $M \in cl(\mathbb{R}^d)$ and Ω^* is the collection of all partition of the interval I, containing a fixed division point $\gamma \in I$. Then*

$$W_\alpha^\beta(M) = \bigcap_{\omega\in\Omega^*} S(M, \omega).$$

Lemma 3. *(*on the semigroup property of the alternating integral*). Let $M \in cl(\mathbb{R}^d)$ and $\gamma \in I$. Then $W_\gamma^\beta(W_\alpha^\gamma(M)) = W_\alpha^\beta(M)$.*

Proof. Let ω', ω'' be arbitrary partitions of the interval $[\alpha, \gamma]$ and $[\gamma, \beta]$ correspondingly. Then $\omega = \omega' \cup \omega'' \in \Omega^*$. It is obvious that each partition $\omega \in \Omega^*$ has such form. Therefore Corollary 1 implies

$$W_\alpha^\beta(M) = \bigcap_{\omega\in\ \Omega^*} S(M, \omega) = \bigcap_{\omega''}\bigcap_{\omega'} S(S(M, \omega'), \omega'')$$

that is why

$$W_\alpha^\beta(M) \subset \bigcap_{\omega''}\bigcap_k S(S(M, \omega_k), \omega''),$$

where $\{\omega_k\}$ is sequence of partitions of the segment $[\alpha, \gamma]$ decreasing with respect to inclusion order. Applying Lemmas 1 and 2 to the left side of last relation, we have

$$W_\alpha^\beta(M) \subset \bigcap_{\omega''} S(\bigcap_k S(M, \omega_k)) = \bigcap_{\omega''} S(W_\alpha^\gamma(M), \omega'') = W_\gamma^\beta(W_\alpha^\gamma(M)).$$

From other side,

$$W_\gamma^\beta(W_\alpha^\gamma(M)) = \bigcap_{\omega''} S(\bigcap_{\omega'} S(M, \omega')) \subset \bigcap_{\omega''} \bigcap_{\omega'} S(S(M, \omega'), \omega'') =$$

$$= \bigcap_{\omega \in \ \Omega^*} S(M, \omega) = W_\alpha^\beta(M).$$

Lemma 3 is proved.

It should be noted that the paper (Pshenichniy and Sagaydak, 1970) contains proof of the semigroup property for the other operator $\widetilde{T}_t$ that is based on rational partitions of the time interval.

Theorem 1. *Let $M \in cl(\mathbb{R}^d)$. Then the following recurrent relation is hold:*

$$W^\tau(M) = \bigcap_{\varepsilon > 0} \bigcap_{\upsilon(\cdot) \in Q(\tau - \varepsilon, \tau)} \left[W^{\tau - \varepsilon}(M) + \int_{\tau - \varepsilon}^{\tau} F(t, \upsilon(t)) dt \right]. \tag{5}$$

Proof. Let ε be an arbitrary number from the interval $(0, \tau)$ and $\Omega^{(\varepsilon)}$ be the collection of partitions ω of the interval $[0, \tau]$ such that $\tau - \varepsilon$ services as a division point. It is evident

$$W^\tau(M) \subset \bigcap_{\omega \in \Omega^{(\varepsilon)}} S(\omega).$$

Further let ω be an arbitrary partition from $\Omega^{(\varepsilon)}$, such that $\omega = \{0 = \tau_0 < \tau_1 < ... < \tau_{l-1} < \tau_l = \tau - \varepsilon < \tau_{l+1} < ... < \tau_{n-1} < \tau\}$. Expressing $S(\omega)$ via S^{n-2} we obtain

$$S(\omega) = \bigcap_{\upsilon_n(\cdot)} \left\{ \bigcap_{\upsilon_{n-1}(\cdot)} \left[S^{n-2} + \int_{n-1} F(t, \upsilon_{n-1}(t)) dt \right] + \int_n F(t, \upsilon_n(t)) dt \right\},$$

where $\upsilon_k(\cdot)$ is an arbitrary element of the collection $Q[\Delta_k]$. Thus By virtue of (3)

$$S(\omega) \subset \bigcap_{\upsilon_n(\cdot)} \bigcap_{\upsilon_{n-1}(\cdot)} \left[S^{n-2} + \int_{\Delta_{n-1} \cup \Delta_n} F(t, \bar{\upsilon}(t)) dt \right] =$$

$$= \bigcap_{\bar{\upsilon}(\cdot)} \left[S^{n-2} + \int_{\Delta_{n-1} \cup \Delta_n} F(t, \bar{\upsilon}(t)) dt \right],$$

where $\bar{\upsilon}(t) = \upsilon_{n-1}(t)$ for $t \in \Delta_{n-1}$ and $\bar{\upsilon}(t) = \upsilon_n(t)$ for $t \in \Delta_n = (\tau_{n-1}, \tau]$.

Continuing such kind of arguments gives the following relation

$$S(\omega) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau - \varepsilon, \tau]} \left[S^l + \int_{\tau - \varepsilon}^{\tau} F(t, \upsilon(t)) dt \right].$$

(here $\tau_l = \tau - \varepsilon$). Therefore

$$\bigcap_{\omega \in \Omega^{(\varepsilon)}} S(\omega) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \bigcap_{\omega'} \left[S(\omega') + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right],$$

where the inner intersection is taken over all partitions ω' of the segment $[0, \tau - \varepsilon]$. Let ω_k be a sequence of partitions of the segment $[0, \tau - \varepsilon]$ decreasing monotonically by inclusion (i.e. $\omega_k \subset \omega_{k+1}$), $\mid \omega_k \mid \to 0$ for $k \to \infty$. Then

$$W^{\tau}(M) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \bigcap_{k} \left[S(\omega_k) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right], \tag{6}$$

Now applying Lemma 2 to the right side of the inclusion (6), we obtain

$$W^{\tau}(M) \subset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right].$$

Since the number ε was arbitrary, we can conclude

$$W^{\tau}(M) \subset \bigcap_{\varepsilon > 0} \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right].$$

Now we are to prove the inverse inclusion. For that it is enough to show

$$S(M + 2\lambda\varepsilon H, \omega) \supset \bigcap_{\upsilon(\cdot) \in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t, \upsilon(t))dt \right]$$

for any $\omega \in \Omega$ and $\varepsilon \in (0, \tau)$.

Let us choose an arbitrary partition $\omega \in \Omega$ and a point $\varepsilon \in (0, \tau)$. It can be considered $\tau - \varepsilon \in [\tau_{l-1}, \tau_l)$ for some l.

First we note

$$S^0(M + 2\lambda\varepsilon H) = M + 2\lambda\varepsilon H = S^0(M) + 2\lambda\varepsilon H.$$

Further taking an arbitrary element $\upsilon_i(\cdot)$ of the collection $Q[\Delta_i]$ suppose $S^i(M + 2\lambda\varepsilon H) \supset S^i(M) + 2\lambda\varepsilon H$. Then

$$S^{i+1}(M + 2\lambda\varepsilon H) = \bigcap_{\upsilon_{i+1}(\cdot)} \left[S^i(M + 2\lambda\varepsilon H) + \int_{i+1} F(t, \upsilon_{i+1}(t))dt \right] \supset$$

$$\supset \bigcap_{\upsilon_{i+1}(\cdot)} \left[\left[S^i(M) + \int_{i+1} F(t, \upsilon_{i+1}(t))dt \right] + 2\lambda\varepsilon H \right] \supset S^{i+1}(M) + 2\lambda\varepsilon H. \tag{7}$$

Let $\int_{\alpha}^{\beta} F(t, \upsilon(t))dt = F_{\alpha}^{\beta}$ for the bbrevity. By virtue of the inclusion(7) we have

$$S^l(M + 2\lambda\varepsilon H) = \bigcap_{\upsilon_l(\cdot)} \left[S^{l-1}(M + 2\lambda\varepsilon H) + F_{\tau_{l-1}}^{\tau_l} \right] \supset$$

$$\supset \bigcap_{\upsilon_l(\cdot)} \left[S^{l-1}(M) + F^{\tau_l}_{\tau_{l-1}} + 2\lambda\varepsilon H\right]. \tag{8}$$

Now we'll estimate the last intersection. The relation (3) implies

$$\bigcap_{\upsilon_l(\cdot)} [S^{l-1}(M) + F^{\tau_l}_{\tau_{l-1}} + 2\lambda\varepsilon H] \supset$$

$$\supset \bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau_l]} [\bigcap_{\upsilon(\cdot)\in Q[\tau_{l-1},\tau-\varepsilon]} [S^{l-1}(M) + F^{\tau-\varepsilon}_{\tau_{l-1}}] + F^{\tau_l}_{\tau-\varepsilon} + 2\lambda\varepsilon H].$$

Noticing

$$\bigcap_{\upsilon(\cdot)\in Q[\tau_{l-1},\tau-\varepsilon]} [S^{l-1}(M) + F^{\tau-\varepsilon}_{\tau_{l-1}}] \supset W^{\tau-\varepsilon}(M)$$

and $\lambda\varepsilon H \supset F^{\tau}_{\tau-\varepsilon}$ we get

$$\bigcap_{\upsilon_l(\cdot)} [S^{l-1}(M) + F^{\tau_l}_{\tau_{l-1}} + 2\lambda\varepsilon H] \supset$$

$$\supset \bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau_l]} [\bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau]} [W^{\tau-\varepsilon}(M) + F^{\tau}_{\tau-\varepsilon}] + F^{\tau_l}_{\tau-\varepsilon} + \lambda\varepsilon H].$$

Let $Y(\varepsilon)$ denotes

$$\bigcap_{\upsilon(\cdot)\in Q[\tau-\varepsilon,\tau]} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t,\upsilon(t))dt\right].$$

Taking into account $0 \in F^{\tau_l}_{\tau-\varepsilon} + \lambda(\tau_l - \tau + \varepsilon)H$ (because $F^{\beta}_{\alpha} \subset \lambda(\beta-\alpha)H$ by definition of λ), and inclusion (8), we have $S^l(M + 2\lambda\varepsilon H) \supset \supset Y(\varepsilon) + \lambda(\tau - \tau_l)H$.

Being repeated such considerations give $S^n(M + 2\lambda\varepsilon H) \supset Y(\varepsilon) + \lambda(\tau - \tau_n)H = Y(\varepsilon)$. Taking the intersection on ε and applying Lemma 1, we have

$$S(\omega) \supset \bigcap_{\varepsilon>0} \bigcap_{\upsilon(\cdot)\in Q(\tau-\varepsilon,\tau)} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t,\upsilon)dt\right]$$

that follows

$$W^{\tau}(M) \supset \bigcap_{\varepsilon>0} \bigcap_{\upsilon(\cdot)\in Q(\tau-\varepsilon,\tau)} \left[W^{\tau-\varepsilon}(M) + \int_{\tau-\varepsilon}^{\tau} F(t,\upsilon)dt\right].$$

The proof of Theorem 1 is finished.

Note that Theorem 1 allows to solve the problem of reducing system (1) from any state $z_0 \in W^{\tau}(M)$ to the state $z(\tau) \in M$ in the same way as the alternating integral in linear differential games of pursuit ((Pontryagin, 1967)-(Pontryagin, 1980)).

3. Simplified schemes for constructing of the Pontryagin alternating integral

of The alternating integral for linear games usually defines using the operation of integrating of multivalued function, that is equivalent to composition of the set of integrals of measurable selections. In the case of quasilinear differential games the second way is applicable only (Mishchenko and Satimov (1974)). Moreover one has to apply the operation of intersection of the family of sets depending on functions instead of more simple operation of geometrical difference as well (compare (Azamov, 1982) with the formula (2)). So a problem of simplification appears: 'Is it possible to use more simple operations in the definition of the alternating integral for quasilinear differential games?'

The first simplified scheme for constructing of the alternating integral was suggested (Nikolskiy, 1985). Its main idea was developed in (Azamov, 1988). Here we give results according to the system (1).

Let $\alpha(\delta)$ be the modules of continuity of $F(t,\upsilon)$, and let $\omega \in \Omega$. Define $L^0 = M$ and

$$L^i = \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} L^{i-1} + 2\alpha(\delta_i)H + F(t,\upsilon)\right] dt L(\omega) = L^n, L^\tau(M) = \bigcap_\omega L(\omega). \quad (9)$$

The formula (9) is a generalization of the simplified scheme of M.S.Nikolskii to the considering case.

Theorem 2. *Let $M \in Ccl(\mathbb{R}^d)$, then*

$$W^\tau(M) = L^\tau(M).$$

Proof. For a convex and closed set L easily can be chec ked the relation

$$\bigcap_{\upsilon(\cdot)\in Q(\Delta_i)} \left[L + \int_i F(t,\upsilon(t))dt\right] \subset \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} L + 2\alpha(\delta_i)H + F(t,\upsilon)\right] dt. \quad (10)$$

Taking $\xi_i \in \Delta_i$ by the definition of the modules of continuity for $F(t,\upsilon)$, one obtains

$$\bigcap_{\upsilon(\cdot)\in Q(\Delta_i)} \left[L + \int_i F(t,\upsilon(t))dt\right] \subset \bigcap_{\upsilon\in Q} \left[L + \int_i F(t,\upsilon)dt\right] \subset$$

$$\subset \bigcap_{\upsilon\in Q} [L + \delta_i\alpha(\delta_i)H + F(\xi_i,\upsilon)\delta_i]. \quad (11)$$

Integrating both parts of the inclusion

$$\bigcap_{\upsilon\in Q} [\frac{1}{\delta_i} L + \alpha(\delta_i)H + F(\xi_i,\upsilon)] \subset \bigcap_{\upsilon\in Q} [\frac{1}{\delta_i} L + 2\alpha(\delta_i)H + F(t,\upsilon)]$$

gives

$$\bigcap_{\upsilon\in Q} [L + \delta_i\alpha(\delta_i)H + F(\xi_i,\upsilon)\delta_i] \subset \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} L + 2\alpha(\delta_i)H + F(t,\upsilon)\right] dt. \quad (12)$$

Relations (11) and (12) imply the inclusion (10). If instead of L consider consequantly S^i, $i = 0, ..., n-1$, one comes to the inclusion $S(\omega) \subset L(\omega)$. Hence,

$$W^\tau(M) \subset L^\tau(M). \tag{13}$$

Further, it is obvious,

$$L^1 \subset \bigcap_{\upsilon(\cdot)\in Q(\Delta_1)} \left[L^0 + 2\delta_1\alpha(\delta_1)H + \int_1 F(t, \upsilon(t))dt\right] \subset$$

$$\subset S^1(M + 2\delta_1\alpha(\delta_1)H),$$

$$L^2 \subset \bigcap_{\upsilon(\cdot)\in Q(\Delta_2)} \left[L^1 + 2\delta_2\alpha(\delta_2)H + \int_2 F(t, \upsilon(t))dt\right] \subset$$

$$\subset S^2(M + 2\sum_{i=1}^{2}\delta_i\alpha(\delta_i)H).$$

Repeating such estimations we get

$$L(\omega) \subset S(M + 2\sum_{i=1}^{n}\delta_i\alpha(\delta_i)H, \omega).$$

Since $\alpha(\delta_i) \leq \alpha(|\omega|)$, then

$$L^\tau(M) \subset \bigcap_{\omega} S(M + 2\tau\alpha(|\omega|)H, \omega).$$

Lemmas 1 2 imply

$$\bigcap_{\omega} S(M + 2\tau\alpha(|\omega|)H, \omega) = W^\tau(M).$$

Hence,

$$L^\tau(M) \subset W^\tau(M). \tag{14}$$

Theorem 2 is proved.

Let us to take note of difference between schemes (2) and (9). The partial sum L^i was being constructed from L^{i-1} applying on each step the additional summand $2\alpha(\delta_i)H$ called "M.S.Nikolsky's cap". If one omits such 'caps' when L^i is constructed, then the inclusion $L^\tau(M) \subset W^\tau(M)$ stays valid but inverse may be not hold (Azamov and Yahshimov, 2000).

Further we describe more schemes for constructing of the alternating integral, using "Nikolskii' caps" by some other way. By $\Phi(\Delta, D)$ we denote the collection of all measurable closed valued mappings $A(\cdot) : \Delta \to cl(\mathbb{R}^d)$, satisfying the condition $\int_\Delta A(t)dt \subset D$. (About the definition of a measurable multivalued mapping see (Ioffe and Tihomirov, 1974).)

Let

$$C^0 = M, C^i = \bigcup_{A(\cdot)} \int_i \bigcap_{\upsilon\in Q} [A(t) + F(t, \upsilon)]dt,$$

where the union is taken over all $A(\cdot) \in \Phi(\Delta_i, C^{i-1} + 2\delta_i\alpha(\delta_i)H)$,

$$C(\omega) = C^n, C^\tau(M) = \bigcap_{\omega} C(\omega). \tag{15}$$

Corollary 2. *Let $M \in Ccl(\mathbb{R}^d)$, then*

$$W^\tau(M) = C^\tau(M).$$

The inclusion $C^\tau(M) \subset W^\tau(M)$ can be proved in the same way as (14) while its inverse $W^\tau(M) = L^\tau(M) \subset C^\tau(M)$ is obvious.

The following scheme was proposed in (Satimov and Karabaev, 1986) for linear differential games. It combines elements of first and second direct Pontryagin methods (Pontryagin, 1967), (Pontryagin, 1980).

For $\omega \in \Omega$ define $B^0 = M$, and

$$B^i = \bigcup_{A(\cdot)} \int_i \bigcap_{v \in Q} [A(t) + F(t,v)]dt,$$

where the union is taken over all $A(\cdot) \in \Phi(\Delta_i, B^{i-1})$. Let

$$B(\omega) = B^n, \quad B^\tau(M) = \bigcup_\omega B(\omega).$$

Lemma 4. *If $M \in cl(\mathbb{R}^d)$, then $B^\tau(M) \subset W^\tau(M)$.*

Proof. Let X be an arbitrary subset of $\mathbb{R}^d$ and $A(\cdot) \in \Phi([\gamma,\theta], X)$, $[\gamma,\theta] \subset I$. We'll use the notation

$$B_\gamma^\theta = \int_\gamma^\theta \bigcap_{v \in Q} [A(t) + F(t,v)]\, dt.$$

First let use prove

$$\bigcup_{A(\cdot) \in \Phi([\gamma,\theta],X)} B_\gamma^\theta \subset W_\gamma^\theta(X), \tag{16}$$

where $W_\gamma^\theta(X)$ is an alternating integral (see part 2).

As before take a partition $\omega = \{\gamma = t_0 < t_1 < t_2 < ... < t_m = \theta\}$ be a partition of segment $[\gamma,\theta]$. Obviously

$$B_{t_{j-1}}^{t_j} = \int_j \bigcap_{v \in Q} [A(t) + F(t,v)]\, dt \subset \bigcap_{v_j(\cdot)} \left[\int_j A(t)dt + \int_j F(t, v_j(t))dt \right],$$

where $v_j(\cdot) \in Q(\Delta_j)$. Using the inclusion (3), we obtain

$$B_\gamma^{t_2} = B_\gamma^{t_1} + B_{t_1}^{t_2} \subset \bigcap_{v_1(\cdot)} \left[\int_\gamma^{t_1} A(t)dt + \int_1 F(t, v_1(t))dt \right] +$$

$$+ \bigcap_{v_2(\cdot)} \left[\int_{t_1}^{t_2} A(t)dt + \int_2 F(t, v_2(t))dt \right] \subset$$

$$\subset \bigcap_{v_2(\cdot)} \left[\bigcap_{v_1(\cdot)} \left[\int_\gamma^{t_2} A(t)dt + \int_1 F(t, v_1(t))dt \right] + \int_2 F(t, v_2(t))dt \right] = S^2 \left(\int_\gamma^{t_2} A(t)dt \right).$$

The continuation of these objections bring us to the following inclusion

$$B_\gamma^{t_m} \subset S^m \left(\int_\gamma^{t_m} A(t)dt \right).$$

Taking into account $t_m = \theta$ and $\int_\gamma^\theta A(t)dt \subset X$, we have $B_\gamma^\theta \subset S(X,\omega)$. Since ω was an arbitrary partition and $A(\cdot) \in \Phi([\alpha,\beta],X)$, then

$$\bigcup_{A(\cdot)\in\Phi([\gamma,\theta],X)} B_\gamma^\theta \subset W_\gamma^\theta(X).$$

As a conclusion of (16) and Lemma 3

$$B^1 \subset W_0^{\tau_1}(M),\ B^2 \subset W_{\tau_1}^{\tau_2}(B^1) \subset W_{\tau_1}^{\tau_2}(W_0^{\tau_1}(M)) \subset W_0^{\tau_2}(M),$$

.. $B^n \subset W_0^{\tau_n}(M) = W^\tau(M)$ that follows $B^\tau(M) \subset W^\tau(M)$. Lemma 4 is proved.

Now, we describe the schemes, where the final "cap" puts on the terminal set only.

Once more for $\omega \in \Omega$ define partial sums of the next scheme as $D^0 = M$,

$$D^i = \int_i \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_i} D^{i-1} + F(t,\upsilon)\right] dt,\ D(\omega) = D^n,\ D^\tau(M) = \bigcup_\omega D(\omega).$$

Theorem 3. *If $M \in cl(\mathbb{R}^d)$ then*

$$W^\tau(M) = \bigcap_{\varepsilon>0} D^\tau(M + \varepsilon H). \tag{17}$$

Proof. Let ε be chosen arbitrarily at the same time let $\omega \in \Omega$ be a partition, possessing the property $\alpha(|\omega|) < \varepsilon/2\tau$. Then by virtue of (10) the following relations hold

$$S^1 \subset \int_1 \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_1}(M + 2\delta_1\alpha(\delta_1)H) + F(t,\upsilon)\right] dt = D^1(M + 2\delta_1\alpha(\delta_1)H),$$

$$S^2 \subset \int_2 \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_2}(D^1(M + 2\delta_1\alpha(\delta_1)H) + 2\alpha(\delta_2)H) + F(t,\upsilon)\right] dt \subset$$

$$\subset \int_2 \bigcap_{\upsilon\in Q} \left[\frac{1}{\delta_2}(D^1(M + 2\sum_{i=1}^2 \delta_i\alpha(\delta_i)H)) + F(t,\upsilon)\right] dt =$$

$$= D^2(M + 2\sum_{i=1}^2 \delta_i\alpha(\delta_i)H),$$

and so forth

$$S^n \subset D^n(M + 2\sum_{i=1}^n \delta_i\alpha(\delta_i)H).$$

As $\alpha(\delta_i) < \alpha(|\omega|) < \varepsilon/2\tau$, we have

$$W^\tau(M) \subset S(M,\omega) \subset D(M + \varepsilon H,\omega) \subset D^\tau(M + \varepsilon H).$$

Since the number $\varepsilon > 0$ was arbitrary, then

$$W^\tau(M) \subset \bigcap_{\varepsilon>0} D^\tau(M + \varepsilon H).$$

Inversely, Lemma 4 implies

$$D^\tau(M+\varepsilon H) \subset W^\tau(M+\varepsilon H).$$

Thus

$$\bigcap_{\varepsilon>0} D^\tau(M+\varepsilon H) \subset \bigcap_{\varepsilon>0} W^\tau(M+\varepsilon H).$$

Finally applying Lemma 1, we obtain

$$\bigcap_{\varepsilon>0} D^\tau(M+\varepsilon H) \subset \bigcap_{\varepsilon>0} W^\tau(M+\varepsilon H) = W^\tau(M).$$

Theorem 3 is also established.

Theorem 4. *Let $M \in Ccl(\mathbb{R}^d)$. Then*

$$W^\tau(M) = \bigcap_{\varepsilon>0} B^\tau(M+\varepsilon H). \tag{18}$$

Proof of the Theorem and its game-theoretical interpretation are similar to the ones for linear case (Azamov, 1982).

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A Competition in the Logistics Market

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Abstract We consider the cases of different number of logistics firms in the market which provide service for customers. The game-theoretic model of choosing order service is constructed. The model is a n-person game with perfect information where clients defined as players. We find equilibrium strategies for clients. The existence of these equilibria is proved.

Keywords: selecting problem, logistics schemes, n-person game with full information, Nash equilibrium, fully-mixed strategies.

1. Introduction

In the modern world the role of competition in the logistics market is very significant. Thus it is logical buyers desire to purchase goods or obtain services at the lowest price in the shortest time. Therefor an important role in the selection patterns of production and distribution of goods and services is the process of selecting the best service option. In this paper we will consider three company which services to build customer orders and provides various ways to make orders. Customers, in turn, refer to the company for the service, while trying to minimize the total cost of implementing the order. At the same time customers are players competing for the best option of receiving the service. There are many publications that address the selecting problems in terms of economic analysis, inventory control theory, queuing theory, statistical evaluation, network planning and management, among which we can provide (Daganzo, 1996, Langevin and Riopel, 2005, Medonza and Ventura, 2009). In (Linke et al., 2002) the problems of the world and the main challenges of such systems, set major tasks for development of the industry are studied. Practical interest in the model presented in (Ghiani et al., 2004, Nooper and Hompel, 2009).

2. The main model

Consider the logistics market with three firms transporting goods for the customers. Each firm defines its own pricing scheme(let firms 1 and 3 serve customers in turn, firm 2 serves all customers together without queue). Customers choose firm trying to minimize net value of service casualties. The game-theoretic approach used to find optimal behavior of customers considered as players.

Denote by τ_1, τ_2, τ_3 - the time of stating in system client in selecting the firm 1 , 2 or 3, respectively, so

$$\tau_1 = \tau_{11} + \tau_{12},$$

where τ_{11} - waiting time of the order by firm 1, τ_{12} - the service time by firm 1;

$$\tau_2 = \tau_{22},$$

as waiting time of service at the firm 2 is zero, where τ_{22} - the service time by firm 2;

$$\tau_3 = \tau_{31} + \tau_{32},$$

where τ_{31} - waiting time of the order by firm 3; τ_{32} - the service time by firm 3.

The parameters τ_1, τ_2, τ_3 are random variables. Define the cost to the customer service by each firms.

Let c_1 - the cost of customer order fulfilment by firm 1, it is fixed and does not depend on the duration of the order the customer. Assume further that c_2 - the cost of customer order fulfilment by firm 2, depending on the duration of customer service by firm 2: $c_2 = c_{21} + c_{22}\tau_{22}$, where c_{21} - fixed price charged for customer order, c_{22} - the cost per unit time customer service by firm 2. And finally, c_3 - the cost of customer order fulfilment by firm 3, $c_3 = c_{32}$, where $c_{31} = 0$ - fixed price charged for customer order is equal to zero, c_{32} - the cost per unit time customer service by firm 3. In addition to the cost of order customers have losses associated with waiting for the order. Let r - specific losses incurred by the client while waiting for the order, then we can determine the total loss associated with the expectation of the order by firm 1, 2 or 3, which will be determined by the following formulas:

$$r\tau_1 = r(\tau_{11} + \tau_{12}),$$

$$r\tau_2 = r\tau_{22},$$

$$r\tau_3 = r(\tau_{31} + \tau_{32}).$$

Now it is possible to calculate the full loss of clients to service devices 1 and 2, respectively:

$$\tilde{Q}_1 = r\tau_1 + c_1,$$

$$\tilde{Q}_2 = (r + c_{22})\tau_{22} + c_{21},$$

$$\tilde{Q}_3 = r\tau_{31} + (r + c_{32})\tau_{32}.$$

Then the average loss of customers for services provided by different firms are determined by the following expectations:

$$Q_1 = E\tilde{Q}_1 = r(E\tau_{11} + E\tau_{12}) + c_1,$$

$$Q_2 = E\tilde{Q}_2 = (r + c_{22})E\tau_{22} + c_{21},$$

$$\tilde{Q}_3 = rE\tau_{31} + (r + c_{32})E\tau_{32}.$$

The problem of the system with two service devices each of which establishes its own order of service was considered in (Bure, 2002) with some adjustment.

Duration of the customer service by the firm 1, 2 and 3 are independent random variables with densities functions:

$$f_1(t) = \frac{1}{\mu_1} e^{-\frac{1}{\mu_1} t}, \ \ t > 0,$$

$$f_2(t) = \frac{1}{\mu_2} e^{-\frac{1}{\mu_2} t}, \ \ t > 0,$$

$$f_3(t) = \frac{1}{\mu_3} e^{-\frac{1}{\mu_3} t}, \ \ t > 0.$$

Assume that at the point of time group of n customers comes to service . It is known that in the service of the firm 1 are k_1 customers (of which $k_1 - 1$ are in line to order fulfillment), in the service of the firm 3 are k_3 customers (of which $k_3 - 1$ are in line to order fulfillment). Each client decides which device to choose for the ordering fulfillment. Let p_i - the probability that the client i chooses device 1, $1 - p_i$ - that the client i chooses device 2.

This model leads to the n-person game, in which customers are the players who choose the order device to implement the order.

3. The game

Define the non-antagonistic game in normal form according
(Petrosyan et al., 1998):
$\Gamma =< N, \{p_i^j\}_{i \in N}, \{H_i\}_{i \in N} >$, where
$N = \{1, \dots, n\}$ - set of players,
$\{p_i^{(j)}\}_{i \in N}$ - set of strategies, $p_i^{(j)} \in [0, 1]$, $j = 1, 2, 3$,
$\{H_i\}_{i \in N}$ - set of payoff functions.

$$H_i = -(p_i^{(1)} Q_{1i} + (1 - p_i^{(1)} - p_i^{(3)}) Q_{2i} + p_i^{(3)} Q_{3i}) \\ = -(p_i^{(1)} (Q_{1i} - Q_{2i}) + p_i^{(3)} (Q_{3i} - Q_{2i}) + Q_{2i}),$$

where $p_i^{(1)}$ is the probability of player i choose firm 1, $p_i^{(3)}$ - is the probability of player i choose firm 3, $p_i^{(2)} = 1 - p_i^{(1)} - p_i^{(3)}$ - is the probability of player i choose firm 2. We consider the casualty functions below: $h_i = -H_i$, $i = 1, \dots, n$.

Define customer specific loss of waiting service r.

$Q_{1i} = r(t_i^{(11)} + t_i^{(12)}) + c_1$ - player i expected loss for firm 1's service, where $t_i^{(11)}$ - mean time of waiting service by firm 1, $t_i^{(12)}$ - mean time of service by firm 1.

$Q_{2i} = (r + c_{22}) t_i^{(22)} + c_{21}$ - player i expected loss for firm 2's service, where $t_i^{(22)}$ - mean time of service by firm 2.

$Q_{3i} = r t_i^{(31)} + (r + c_{32}) t_i^{(32)}$ - player i expected loss for firm 3's service, where $t_i^{(31)}$ - mean time of waiting service by firm 3, $t_i^{(32)}$ - mean time of service by firm 3.

Firms' service times are independent exponential distributed random variables with density functions $f_1(t) = \frac{1}{\mu_1} e^{\frac{1}{\mu_1} t}$, $f_2(t) = \frac{1}{\mu_2} e^{\frac{1}{\mu_2} t}$, $f_3(t) = \frac{1}{\mu_2} e^{\frac{1}{\mu_3} t}$, $(t > 0)$ respectively.

Customers choose only one of three logistic firms. There are k_1 customers on service in the firm $1(k_1 - 1$ of them are in the queue) and k_3 customers on service in the firm $3(k_3 - 1$ of them are in the queue).

4. Main results. The point of equilibrium

Theorem 1. *There exists a unique point of equilibrium* $(p_1, \dots, p_n)$, $i = 1, \dots, n$ *in the game* Γ *defined as follows:*

1) the pure strategies $p_i = (1, 0, 0)$,$i = 1, \dots, n$, *if:*

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1<\mu_2(r+c_{22})+c_{21},$$

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1<\mu_3(r(k_3+1)+c_{32}).$$

2) the pure strategies $p_i=(0,1,0)$,$i=1,\ldots,n$,
if:

$$\mu_2(r+c_{22})+c_{21}<\mu_3(r(k_3+1)+c_{32}),$$

$$\mu_2(r+c_{22})+c_{21}<\mu_1(r(k_1+1))+c_1.$$

3) the pure strategies $p_i=(0,0,1)$,$i=1,\ldots,n$,
if:

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})<\mu_1(r(k_1+1))+c_1,$$

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})<\mu_2(r+c_{22})+c_{21}.$$

4) the fully-mixed strategies under the choice of two firms
$p_i=\Big(\frac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)},1-$
$\frac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)},0\Big)$,
$i=1,\ldots,n$,
if:

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1\le\mu_3(r(k_3+1)+c_{32}),$$

$$\mu_1(r(k_1+1))+c_1\le\mu_2(r+c_{22})+c_{21}\le\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1.$$

5) the fully-mixed strategies under the choice of two firms
$p_i=\Big(0,1-\frac{\mu_2(r+c_{22})+c_{21}-\mu_3(r(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)},$
$\frac{\mu_2(r+c_{22})+c_{21}-\mu_3(r_3(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)}\Big)$,
$i=1,\ldots,n$,
if:

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})\le\mu_1(r(k_1+1))+c_1,$$

$$\mu_3(r(k_3+1)+c_{32})\le\mu_2(r+c_{22})+c_{21}\le\mu_3(r(k_3+1)r+\frac{1}{2}r(n-1)+c_{32}).$$

6) the fully-mixed strategies under the choice of two firms
$p_i=\Big(\frac{\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}(n-1)(\mu_1 r-\mu_3 r)},0,$
$1-\frac{\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}(n-1)(\mu_1 r-\mu_3 r)}\Big)$,
$i=1,\ldots,n$,
if:

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1\le\mu_2(r+c_{22})+c_{21},$$

$$\mu_1(r(k_1+1))+c_1 \le \mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}),$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1,$$

or

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}) \le \mu_2(r+c_{22})+c_{21},$$

$$\mu_1(r(k_1+1))+c_1 \le \mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}),$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1.$$

7) the fully-mixed strategies

$$p_i = \Big(\frac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)},$$

$$1-\frac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)}-$$

$$\frac{\mu_2(r+c_{22})+c_{21}-\mu_3(r(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)}, \frac{\mu_2(r+c_{22})+c_{21}-\mu_3(r(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)}\Big)$$

,

$i = 1, \ldots, n,$
if:

$$\mu_1(r(k_1+1))+c_1 \le \mu_2(r+c_{22})+c_{21} \le \mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1),$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_2(r+c_{22})+c_{21} \le \mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}),$$

$$\mu_1(r(k_1+1))+c_1 \le \mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}),$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1.$$

Proof. If m players including the player i choose firm 1, then player i occupy any of m places in line for service from the firm 1 with probability $\frac{1}{m}$ according (Bure, 2002). Conditional expectation waiting time before service player i without the service time players already in service by firm 1, provided that l players of the m proceed player i:

$$\sum_{l=0}^{m-1} l\mu_1\frac{1}{m} = \frac{1}{m}\mu_1\sum_{l=0}^{m-1} l = \frac{1}{m}\mu_1\frac{m(m-1)}{2} = \frac{1}{2}\mu_1(m-1) \tag{1}$$

The expectation for firm 3 service is defined similarly.

Let $P_r^{(j)}(l)$ be the probability that r players from l choose firm j , $j = 1, 2, 3$. Then we have:

$$\sum_{m=1}^{n}\frac{1}{2}\mu_1(m-1)P_{m-1}^{(1)}(n-1) = \sum_{m=0}^{n-1}\frac{1}{2}\mu_1 m P_m^{(1)}(n-1) \tag{2}$$

This expression for firm 3 is defined similarly.

Now we obtain expression for conditional expectation full service time by firm 1, 2 ,3 respectively:

$$t_i^{(1)} = k_1\mu_1 + \frac{1}{2}\mu_1 \sum_{m=1}^{n}(m-1)P_{m-1}^{(1)}(n-1) + \mu_1 = k_1\mu_1 + \frac{1}{2}\mu_1 \sum_{l=0}^{n-1} lP_l^{(1)}(n-1) + \mu_1,$$

$$t_i^{(2)} = \mu_2,$$

$$t_i^{(3)} = k_3\mu_1 + \frac{1}{2}\mu_3 \sum_{v=1}^{n}(v-1)P_{v-1}^{(3)}(n-1) + \mu_3 = k_3\mu_3 + \frac{1}{2}\mu_3 \sum_{h=0}^{n-1} hP_h^{(3)}(n-1) + \mu_3.$$

Expected loss for firms' 1, 2 and 3 service define as follows:

$$Q_{1i} = r(k_1\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1) + c_1,$$

$$Q_{2i} = (r + c_{22})\mu_2 + c_{21},$$

$$Q_{3i} = \mu_3(r(k_3 + 1) + \frac{1}{2}r \sum_{z\ =1, z\neq i}^{n} p_z + c_{32}).$$

Then the function of expected loss is given by:

$$h_i = -H_i = p_i^{(1)}Q_{1i} + p_i^{(2)}Q_{2i} + p_i^{(3)}Q_{3i} = p_i^{(1)}(Q_{1i} - Q_{2i}) + Q_{2i} + p_i^{(3)}(Q_{3i} - Q_{2i})$$

Consider the following expressions:

$$Q_{1i} - Q_{2i} = \mu_1(r(k_1 + 1) + \frac{1}{2} \sum_{m\ =1, m\neq i}^{n} p_m) - \mu_2(r + c_{22}) - c_{21} + c_1$$

$$Q_{3i} - Q_{2i} = \mu_3(r(k_3 + 1) + \frac{1}{2} \sum_{z\ =1, z\neq i}^{n} p_z + c_{32}) - \mu_2(r_i + c_{22}) - c_{21}$$

$$Q_{3i} - Q_{1i} = \mu_3(r(k_3 + 1) + \frac{1}{2} \sum_{l\ =1, l\neq i}^{n} p_l + c_{32}) - \mu_1(r(k_1 + 1) + \frac{1}{2}(1 - \sum_{l\ =1, l\neq i}^{n} p_l)) - c_1$$

Now we ready to prove that $(p_1^*, \ldots, p_n^*)$ is really the point of equilibrium using (Feller, 1984).

The following situations are possible:

1) (1,0,0), i.e. all players except player i choose only one firm 1, then under conditions

$$Q_1 - Q_2 < 0$$

$$Q_1 - Q_3 < 0$$

player i have to choose the same strategy. So we can write condition for the first case as

$$\mu_1 r((k_1 + 1) + \frac{1}{2}(n-1)) + c_1 < \mu_2(r + c_{22}) + c_{21},$$

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1<\mu_3(r(k_3+1)+c_{32}).$$

2) (0,1,0) i.e. all players except player i choose only one firm 2, then under conditions

$$Q_2-Q_1<0$$

$$Q_2-Q_3<0$$

player i have to choose the same strategy. So we can write condition for the second case as

$$\mu_2(r+c_{22})+c_{21}<\mu_3(r(k_3+1)+c_{32}),$$

$$\mu_2(r+c_{22})+c_{21}<\mu_1(r(k_1+1))+c_1.$$

3) (0,0,1) i.e. all players except player i choose only one firm 3, then under conditions

$$Q_3-Q_3<0$$

$$Q_3-Q_1<0$$

player i have to choose the same strategy. So we can write condition for the third case as

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})<\mu_1(r(k_1+1))+c_1,$$

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})<\mu_2(r+c_{22})+c_{21}.$$

4)$p_i=\Big(\dfrac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)},$
$1-\dfrac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)},0\Big)$,
$i=1,\dots,n,$

i.e. all players except player i choose between firm 1 and firm 2, then under violation of first condition in 1) and second condition in 2) and satisfaction of second condition in 1) player i have to choose the same strategy. We can write this conditions as follows:

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1\le\mu_3(r(k_3+1)+c_{32}),$$

$$\mu_1(r(k_1+1))+c_1\le\mu_2(r+c_{22})+c_{21}\le\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1.$$

5) $p_i=\Big(0,1-\dfrac{\mu_2(r+c_{22})+c_{21}-\mu_3(r(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)},$
$\dfrac{\mu_2(r+c_{22})+c_{21}-\mu_3(r_3(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)}\Big),$
$i=1,\dots,n,$

i.e. all players except player i choose between firm 2 and firm 3, then under violation of first condition in 2) and second condition in 3) and satisfaction of first condition in 3) player i have to choose the same strategy. We can write this conditions as follows:

if:

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})\le\mu_1(r(k_1+1))+c_1,$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_2(r+c_{22})+c_{21} \le \mu_3(r(k_3+1)r+\frac{1}{2}r(n-1)+c_{32}).$$

6) $p_i = \Big(\frac{\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}(n-1)(\mu_1 r-\mu_3 r)}, 0,$
$1-\frac{\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32})-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}(n-1)(\mu_1 r-\mu_3 r)}\Big),$
$i = 1, \ldots, n,$

i.e. all players except player i choose between firm 1 and firm 3, then under violation of second condition in 1) and satisfaction of first condition in 1) or second condition in 3) player i have to choose the same strategy. We can write this conditions as follows:

if:

$$\mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1 \le \mu_2(r+c_{22})+c_{21},$$

$$\mu_1(r(k_1+1))+c_1 \le \mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}),$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1,$$

or

$$\mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}) \le \mu_2(r+c_{22})+c_{21},$$

$$\mu_1(r(k_1+1))+c_1 \le \mu_3(r(k_3+1)+\frac{1}{2}r(n-1)+c_{32}),$$

$$\mu_3(r(k_3+1)+c_{32}) \le \mu_1 r((k_1+1)+\frac{1}{2}(n-1))+c_1.$$

$$7) p_i = \Big(\frac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)},$$
$$1-\frac{\mu_2(r+c_{22})+c_{21}-\mu_1 r(k_1+1)-c_1}{\frac{1}{2}\mu_1 r(n-1)} -$$
$$\frac{\mu_2(r+c_{22})+c_{21}-\mu_3(r(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)},$$
$$\frac{\mu_2(r+c_{22})+c_{21}-\mu_3(r(k_3+1)+c_{32})}{\frac{1}{2}\mu_3 r(n-1)}\Big),$$

i.e. all players except player i choose between all three firms, then under violation of all conditions in 1) - 3) player i have to choose the same strategy.

Lets prove the uniqueness of the equilibrium.

Suppose that probabilities $p_i^{(j)}$ of choosing a firm j can be different $i = 1, \ldots, n$, $j = 1, 2, 3$. Value $\sum\limits_{l=0}^{n-1} l P_l^{(1)}(n-1)$ equals the sum of the expectation of success (a success we mean the choice of firm 1), then we have

$$\sum_{l=0}^{n-1} l P_l^{(1)}(n-1) = \sum_{m\ =1, m\ne i}^{n} p_m,$$

expected loss for firm's 1 service:

$$Q_{1i} = r(k_1\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1) + c_1,$$

Consider the expression:

$$Q_{1i} - Q_{2i} = r(k_1\mu_1 + \frac{1}{2}\mu_1 \sum_{m\ =1, m\neq i}^{n} p_m + \mu_1) + c_1 - (r + c_{22})\mu_2 - c_{21} = 0, \quad (3)$$

assuming the sum of probabilities is unknown value.

If $Q_1 - Q_2 < 0$, then players have to choose $p_i^{(1)} = 1$. If $Q_1 - Q_2 > 0$, then players have to choose $p_i^{(1)} = 0$. If both conditions violated then is uniquely determined by solving the equation (3).

$\sum\limits_{m\ =1, m\neq i}^{n} p_m$ should be equal for all $i = 1, \ldots, n$, then $p_i = p_j,\ i \neq j$.

Case $Q_3 - Q_2$ treated similarly.

Consider the expression:

$$Q_{3i} - Q_{2i} = \mu_3(r(k_3 + 1) + \frac{1}{2} \sum_{z\ =1, z\neq i}^{n} p_z + c_{32}) - \mu_2(r_i + c_{22}) - c_{21} = 0, \quad (4)$$

assuming the sum of probabilities is unknown value.

If $Q_3 - Q_2 < 0$, then players have to choose $p_i^{(3)} = 1$. If $Q_3 - Q_2 > 0$, then players have to choose $p_i^{(3)} = 0$. If both conditions violated then is uniquely determined by solving the equation (4).

$\sum\limits_{z\ =1, z\neq i}^{n} p_z$ should be equal for all $i = 1, \ldots, n$, then $p_i = p_j,\ i \neq j$.

And in the last case $Q_3 - Q_1$ we can assume that under the choice of two firms 3 or 1 each player choose firm 3 with probability p_l then it choose firm 1 with probability $1 - p_l$. Then we have

$$Q_{3i} - Q_{1i} = \mu_3(r(k_3+1) + \frac{1}{2} \sum_{l\ =1, l\neq i}^{n} p_l + c_{32}) - \mu_1(r(k_1+1) + \frac{1}{2}(1 - \sum_{l\ =1, l\neq i}^{n} p_l)) - c_1, \quad (5)$$

If $Q_3 - Q_1 < 0$, then players have to choose $p_i^{(3)} = 1$. If $Q_3 - Q_1 > 0$, then players have to choose $p_i^{(3)} = 0$. If both conditions violated then is uniquely determined by solving the equation (5).

$\sum\limits_{l\ =1, l\neq i}^{n} p_l$ should be equal for all $i = 1, \ldots, n$, then $p_i = p_j,\ i \neq j$.

So the strategies of customers optimal behavior under competition in the logistics market are found.

5. Conclusion

In this paper we consider the market of logistics service where some firms operate. Each firm prefer its own pricing police. Customers seek for service trying to minimize its operational costs. We find the optimal behavior of customers in the logistics market under choice of three firms. The Nash equilibria was found. The existence of such equilibria was prooved.

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Investments in Productivity and Quality under Trade Liberalization: Monopolistic Competition Model *

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Abstract We study impact of trade liberalization on firms productivity and product quality in a monopolistic competition model. Utility has variable elasticity of substitution (VES), a producer can invest in decreasing marginal cost or in increasing quality and free entry drives profits to zero. Then in a closed economy such investments increase with the market size if and only if utility shows increasing "relative love for variety" which is elasticity of the inverse demand. Expanding these findings to international trade setting, we expect to find comparative statics of equilibria with respect to the market size and trade costs.

Keywords: investments, quality, monopolistic competition, trade liberalization, relative love for variety, country size.

1. Introduction

The cross-countries differences in productivity and quality are noticeable and there can be various explanations. Recent empirical papers on international trade show that (1) firms operating in bigger markets have lower markups, see e.g. (Syverson, 2007); (2) firms tend to be larger in larger markets, see e.g. (Campbell and Hopenhayn, 2005); (3) larger economies export higher volumes of each good, export a wider set of goods, and export higher-quality goods, see e.g. (Hummels and Klenow, 2005; (4) within an industry there can be considerable firm's heterogeneity: firms differ in efficiency, in exporting or not (associated with high/low efficiency), in wages, see review in (Reddings, 2011); (5) investment decisions are positively correlated with export status of a firm, see e.g. (Aw et al., 2008).

Modern theoretical explanation of these and other empirical regularities is based on some variations of Krugman's monopolistic competition model (Krugman, 1979)

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and its variant with heterogeneous firms, suggested by (Melitz, 2003). Unfortunately, mainly these findings are based on specific functional form of the preferences, namely utility function with constant elasticity of substitution (CES-function). This functional form has some unavoidable shortcomings broadly criticized in the literature, see e.g. (Ottaviano et al., 2002), (Behrens and Murata, 2007) and (Zhelobodko et al., 2010). Thereby we find it reasonable to put efforts in constructing related theory for utility function with *variable elasticity of substitution*.

To obtain one of possible explanations of regularities (1)-(5) mentioned, this project develops such monopolistic competition models from (Zhelobodko et al., 2010) and (Zhelobodko et al., 2011) towards endogenous choice of technology. The first part of this project models a closed economy in the spirit of (Vives, 2008) (and references within this paper). However, these papers consider partial equilibrium in oligopoly setting, whereas we take general equilibrium of monopolistic competition, which is more suitable for international trade. Our basic question is the impact of market size on equilibrium investment, prices, number of firms in the industry, total investments in the economy. We show below that this model with endogenous investments is equivalent to the model with endogenous quality. Thereby we simultaneously answer on the questions, how the market size influences not only on the mass of varieties, but also on the quality. Main idea is that when a firm sells to 1.5 billion of people (we have in mind China), it has more incentives to invest a fixed amount in decreasing its variable cost or in increasing its quality to exploit economies of scale.

Respectively, when two or more countries decrease the trade barriers, these economies of scale should work in the same direction and respective welfare gains can be as important as the increase in variety of goods (Krugman, 1979) and selection of best producers (Melitz, 2003), and the second, main part of our project is devoted to international trade. How liberalization of trade (decrease in trade costs) and countries size influence the investment decisions, the size of the firms, prices, and the number of firms in the industry? In particular, the countries of different size can choose different production technologies, thereby endogenous cross-country heterogeneity may appear and generate the Ricardian motive for trade. Here most close analogue is the paper (Tanaka, 1995) that considers the Slap's model of circular city and influence of trade liberalization on quality. The main finding is that trade liberalization decreases the product quality. Another paper (Fan, 2005) considers competition in the market of intermediate goods and studies influence of the country size on the quality of goods under specific functional forms of preferences and production functions. Several papers are devoted to influence of trade on technology adoption in the case of heterogeneous firms. In (Bustos, 2011) the classical Melitz model is supplemented by endogenous choice of technology. The set of available technologies is discrete: low marginal costs with high fixed costs or the opposite one. The finding is that trade liberalization increases the share of firms using high-investment technology. In (Yeaple, 2005) the set of technologies is also discrete. Every producer chooses not only technology, but also the quality of labour used (there are two types of the labour, skilled and unskilled). Then exporting firms are larger, employ more advanced technologies, pay higher wages, are more productive, and a reduction in trade frictions can induce firms to switch technologies, leading to an expansion of trade volumes. The main departures of this paper from those described above are: (1) our utility function is of general form; (2) our set of technologies is continuous; (3) we assume Indra-country homogeneity of producers. Of course, homogeneity of producers is a restrictive assumption and the next step would be to model the technology choice model with intra-country heterogeneity, in the spirit of (Zhelobodko et al., 2011) and similar paper (Dhingra and Morrow, 2011) (in both papers technology is fixed).

In Section 2 we introduce our model of closed economy and Section 3 gives related preliminary results. In Section 4 we introduce an open economy model and pose the questions studied.

2. Basic model: closed economy with endogenous technology

We start now modeling the closed economy with endogenous investments in technology, like in (Vives, 2008) but in general equilibrium and monopolistic competition.

Our model departures from the standard Dixit-Stiglitz monopolistic competition model (Dixit and Stiglitz, 1977) to generalize their approach in two directions: the possibility of investment in productivity and quite general utility function. We consider one sector and one production factor interpreted as labour. There are two types of agents: big number L of identical consumers and an endogenous interval $[0, N]$ of identical firms. Each firm produces her own variety of some "aggregate commodity" and the nature of substitution among these varieties determines different outcomes of competition, as we show.

2.1. Consumer

Each consumer maximizes her utility under the budget constraint through choosing an infinite-dimensional consumption vector (integrable function) $X : [0, N] \to R_+$. All consumers behave symmetrically, so consumer's index is redundant. As in (Krugman, 1979), (Vives, 1999) and (Zhelobodko et al., 2011), the preferences are described by the additive-separable utility function maximized under the budget constraint:

$$\max_X \int_0^N u(x_i)di, \qquad s.t. \quad \int_0^N p_i x_i di \leq w + \frac{\int_0^N \pi_i di}{L} = 1.$$

Here N is the endogenous mass of firms or the length of the product line, i.e, the scope (the interval) of varieties. Scalar x_i is "consumption" or amount of $i-th$ variety consumed by each consumer and $X = (x_i)_{i \leq N}$. Elementary utility function $u(\cdot)$ satisfies

$$u(0) = 0,\ u'(x_i) > 0,\ u''(x_i) < 0,\ r_{u'}(x_i) \equiv -\frac{x_i u'''(x_i)}{u''(x_i)} < 2,$$

i.e., it is everywhere increasing, strictly concave and its concavity is restricted to guarantee strict concavity of producer's profit, that ensures equilibria symmetry and uniqueness, see (Zhelobodko et al., 2010). We need not additional restrictions like homothety or CES. Instead, as in (Krugman, 1979), (Vives, 1999) and (Zhelobodko et al., 2011), we use the Arrow-Pratt measure of concavity defined for any function g as

$$r_g(z) = -\frac{zg''(z)}{g'(z)}.$$

Through varying such characteristics for utility and cost functions we classify market outcomes below, because $r_u = 1/\sigma$ is the inverse to elasticity of substitution σ among varieties and to elasticity of demand for each variety (standardly). Moreover, function r_u being the elasticity of inverse demand function, it measures "love for variety" (Zhelobodko et al., 2011).

In the budget constraint w is wage, p_i is the price for x_i and π_i is the profit of $i-th$ firm. So, the income of the each consumer consists of the unit of the labour offered nonelastically,[1] and fare share of total profit in the economy. However, actually $\pi_i = 0$ at the equilibrium due to free-entry condition. Finally, since we consider the general equilibrium model, the price level and wage can be normalized to $w \equiv 1$, therefore income amounts to 1.

The First Order Condition (FOC) for consumer's maximization program, standardly, entails the inverse demand for $i-th$ variety in the form

$$p(x_i, \lambda) = \frac{u'(x_i)}{\lambda}\,. \tag{1}$$

Thus, willingness to pay depends both on quantity of the individual consumption x_i and on marginal utility of income λ and increasing marginal utility of income λ leads to a decrease in demand.

2.2. Producer

On the supply side, we standardly assume one-to-one correspondence: each variety is produced by one firm that produces a single variety. However, unlike classical setting, each producer chooses the technology level. Namely, if she spends f units of labour as fixed costs, then total costs of producing y units of output is $c\,(f)\,y+f$ units of the labour. It is natural to assume decreasing and convex costs:

$$c'(f) < 0, \quad c''(f) > 0, \quad \lim_{f\to\infty} c(f) > c_0 > 0.$$

Thereby more expensive factory would have smaller marginal costs, investment in productivity shows decreasing returns to scale and cost cannot fade to zero, being higher than some positive c_0.[2]

Using the inverse demand function $p(x_s, \lambda)$ from (1), the profit maximization of $s-th$ producer can be formulated as[3]

$$\pi_s\,(x_s, f_s, \lambda) = (p(x_s, \lambda) - c(f_s))\,Lx_s - f_s = \left(\frac{u'(x_s)}{\lambda} - c(f_s)\right) Lx_s - f_s \to \max_{x_s\ge 0, f\ge 0}.$$

Under our assumption about continuum of producers, it is standard to prove that each producer s has a negligible effect on the whole market, i.e. the Lagrange multiplier λ can be treated as (positive) constant by each s. This Lagrange multiplier is the natural aggregate statistic of the market: the bigger is marginal utility of income λ, the smaller is the demand and therefore smaller the profit of producers.

[1] It means that consumer sells her unit of labour under any wages, prices, etc.

[2] This assumption is needed for nonnegative cost function, Second Order Condition (SOC) and existence of maximum in profit maximization.

[3] Standardly, maximization of monopolistic profit w.r.t. price or quantity gives same results.

Thereby this λ can be interpreted as the degree of competition among the producers of differentiated goods, like a price index in standard Dixit-Stiglitz model, see (Zhelobodko et al., 2011).

Each producer maximizes profit w.r.t. supply x and investment f and FOC is

$$\frac{u''(x_s)x_s + u'(x_s)}{\lambda} - c(f_s) = 0, \tag{2}$$

$$c'(f_s)Lx_s + 1 = 0. \tag{3}$$

These equations are valid under SOC:

$$u'''(x_s)x_s + 2u''(x_s) < 0 \Leftrightarrow r_{u'}(x_s) < 2, \tag{4}$$

$$-\frac{(u'''(x_s)x_s + 2u''(x_s))\, c''(f_s)x_s}{\lambda} - (c'(f_s))^2 > 0. \tag{5}$$

Since for each producer the maximization profit problems are the same, further we consider only symmetric equilibria and denote $x_s = x,\ f_s = f\ \forall s$.

2.3. Equilibrium

Entry. Like in standard monopolistic competition model, we assume that firms enter into the market until their profit remains positive. Therefore free entry implies zero-profit condition

$$\frac{u'(x)}{\lambda} - c(f) = \frac{f}{Lx}. \tag{6}$$

Labour balance. Under symmetric equilibrium ($f_i = f$, $x_i = x$) the balance in labour market takes the form

$$\int_0^N (c(f_i)x_iL + f_i)\, di = N\,(c(f)xL + f) = L. \tag{7}$$

Summarizing, we define *symmetric equilibrium* as a bundle $(x^*, p^*, \lambda^*, f^*, N^*)$ satisfying the following relations:

1) rationality in consumption (1);
2) rationality in production (2)-(3) and (4)-(5);
3) free entry condition (6) and balance in labour market (7).

Now we rewrite the equilibrium equations in more convenient form, using the Arrow-Pratt measure of concavity $r_g(x)$ for any function g and excluding λ.

Proposition 1. *Equilibrium consumption/investment couple (x^*, f^*) in one-sector model with endogenous technology is the solution to the system*

$$\frac{r_u(x)\,x}{1 - r_u(x)} = \frac{f}{Lc(f)},$$

$$(1 - r_{\ln c}(f) + r_c(f))\,(1 - r_u(x)) = 1,$$

under the conditions

$$r_u(x) < 1, \qquad (2 - r_{u'}(x))\, r_c(f) > 1.$$

Subsequently equilibrium mass of firms N^* is determined by equation

$$N = \frac{L}{c(f)xL + f}.$$

2.4. Another interpretation: quality

Consider now similar setting where producer chooses investments in quality of production instead of investments in productivity. This setting is shown now to be equivalent, amounting to the same system of equilibrium equations. Therefore, the study of comparative statics of the model with endogenous productivity or endogenous quality is the *same*; these are two similar manifestations of endogenous technology.

The utility of each consumer takes the form (cf. (Tirole, 1988)) $\int_0^N u(q_i x_i)di$, whereq_i is the quality level of $i-th$ variety and x_i is related consumption volume, so that $z_i = q_i x_i$ is the satisfaction volume. Then, from FOC similar to previous one, the inverse demand function for $i-th$ variety with quality q_i is

$$p(x_i, q_i, \lambda) = \frac{q_i u'(q_i x_i)}{\lambda}.$$

Under output $y_i = Lx_i$ the cost function of $i-th$ producer is

$$\tilde{c}(q_i)y_i + \tilde{f}(q_i),$$

where, as before, $\tilde{c}(q_i), \tilde{f}(q_i)$ are marginal and fixed costs respectively. It is natural to assume that both derivatives with respect to q_i are positive, $\tilde{c}'(q_i) > 0, \tilde{f}'(q_i) > 0$ because to increase the quality of the production the producer should spend more labour per unit and also use more expensive technology. Thus, the profit maximization of $i-th$ producer amounts to

$$\left(\frac{q_i u'(q_i x_i)}{\lambda} - \tilde{c}(q_i)\right) Lx_i - \tilde{f}(q_i) \to \max_{x_i \geq 0, q_i \geq 0}.$$

Let us introduce auxiliary variables $f_i = \tilde{f}(q_i)$. Due to monotonicity of function $\tilde{f}$, there exists one-to-one correspondence between quality of production q_i and the value of fixed costs f_i :

$$q_i = q_i(f_i) = \tilde{f}^{-1}(f_i).$$

Besides, we define

$$c\left(f_i\right) = \frac{\tilde{c}(q_i(f_i))}{q_i(f_i)}, \qquad z_i = q_i(f_i)x_i.$$

Then the problem of $i-th$ producer can be rewritten in the following equivalent form:

$$\left(\frac{u'(z_i)}{\lambda} - c\left(f_i\right)\right) Lz_i - f_i \to \max_{z_i \geq 0, f_i \geq 0}.$$

Obviously, this program appears equivalent to the program with endogenous productivity introduced previously. Thus, the model with endogenous quality is mathematically equivalent to the model with endogenous investments, only quantity consumed is measured now in satisfaction z.

3. Preliminary comparative statics for closed economy

Let us study the impact of market size L on the equilibrium variables: prices p, firm size Lx and mass of firms N, investment of each firm f, and total investments in the economy (Nf). For this purpose we consider the system of equilibrium equalities as implicit function connecting (x, f, N) and L. After direct differentiation and long rearrangements we obtain elasticities of main variables (in terms of concavity of basic functions), elasticities signs, and classify the market outcomes according to increasing/decreasing $r_u(x)$ (Increasing Elasticity of the Inverse Demand – IEID or Decreasing Elasticity of the Inverse Demand – DEID).

Proposition 2. *Elasticities of the equilibrium variables w.r.t. market size L in one-sector economy are*

$$\mathcal{E}_x = \frac{(1-r_{\ln c})(1-r_u)}{(2-r_{u'})r_c - 1}, \qquad \mathcal{E}_{Lx} = \frac{r_c r'_u x}{((2-r_{u'})r_c - 1)r_u}$$

$$\mathcal{E}_f = \frac{r'_u x}{((2-r_{u'})r_c - 1)r_u}, \qquad \mathcal{E}_{Nf} = \frac{(1-r_{\ln c})^2(1-r_u)^2}{((2-r_{u'})r_c - 1)r_c} + \frac{1}{r_c} + r_u,$$

$$\mathcal{E}_N = r_u - \frac{(1-r_{\ln c})(1-r_u)^2}{(2-r_{u'})r_c - 1},$$

$$\mathcal{E}_p = -\frac{r_c r'_u x}{(2-r_{u'})r_c - 1}, \qquad \mathcal{E}_{\frac{p-c}{p}} = \frac{(1-r_{\ln c})r'_u x}{(2-r_{u'})r_c - 1},$$

and their signs satisfy classification Table 1:

	DEID		*CES*	*IEID*		
	$r'_u < 0$		$r'_u = 0$	$r'_u > 0$		
	$r_{\ln c} \leq 1$	$r_{\ln c} > 1$	$r_{\ln c} \neq 1$	$r_{\ln c} > 1$	$r_{\ln c} = 1$	$r_{\ln c} < 1$
$\mathcal{E}_x$	$\nexists$	$-$	$-$	$-$	0	$+$
$\mathcal{E}_{Lx}$	$\nexists$	$-$	0	$+$	$+$	$+$
$\mathcal{E}_f$	$\nexists$	$-$	0	$+$	$+$	$+$
$\mathcal{E}_{Nf}$	$\nexists$	$+$	$+$	$+$	$+$	$+$
$\mathcal{E}_N$	$\nexists$	$+$	$+$	$+$	$+$	?
$\mathcal{E}_p$	$\nexists$	$+$	0	$-$	$-$	$-$
$\mathcal{E}_{\frac{p-c}{p}}$	$\nexists$	$+$	0	$-$	0	$+$

In Table 1, $\mathcal{E}_x$ is the elasticity of consumption of one variety, $\mathcal{E}_f$ - is the elasticity of investment per firm, $\mathcal{E}_N$ - the elasticity of mass of firms, $\mathcal{E}_{Lx}$ - the elasticity of total output of one variety, $\mathcal{E}_{Nf}$ - the elasticity of total investment, $\mathcal{E}_p$ - the elasticity of price, $\mathcal{E}_{\frac{p-c}{p}}$ - the elasticity of mark-up (we drop the proofs).

Commenting, we would say that generally these results are rather similar to conclusions in (Vives, 2008) obtained for oligopolistic model. However, our table shows in more details the influence of country/market size on related economy; Vives mainly uncovered the influence of market size on investment decisions. As we can see from the table, there can be five different types of the equilibria. For increasing/decreasing investments, utility characteristic r'_u is the criterion. Namely, standard CES case (implying constant elasticity of demand and degenerate reactions to market size) is the borderline between markets with decreasing (DEID) or increasing (IEID) elasticity of inverse demand. Main finding is that DEID *class show decreasing investments* whereas under IEID *individual investments increase in response to growing market* and investment is always positively correlated with the size of the firm. Decrease of both Lx and f happens under DEID because the mass of firms grows too fastly, excessive competition makes the output shrinking, outweighing the market-size motive to invest in marginal productivity/quality. Nevertheless, total economy investment Nf always grows because growing mass of firms dominates even when f decreases.

As to prices in this model, they respond to market size in the same way as we observed under exogenous technology (f, c) in (Zhelobodko et al., 2010); there prices also increased under DEID but decreased under IEID preferences. This discrepancy got a clear explanation. In all cases increasing market has positive effect on profits of existing firms, that invites new firms into the industry, so N increases. Then growing competition pushes marginal utility of income λ up, so consumption x of each variety decreases. Paradoxically, under IEID too convex inverse demand function (shifting down with growth of λ) pushes the price up in response. Such regularity generally remains valid under our endogenous technology, though in essence it somehow combines comparative statics w.r.t. market size L with comparative statics in costs c and f. Table 1 shows that market size effects on price generally prevail, though consumption and mass of firms may behave in a new fashion in the exotic case ($r_u' > 0$, $r_{\ln c} < 1$).

Interestingly, the nature of the cost function turns out to be the criterion only for increasing/decreasing individual consumption of each variety and related variable called markup $M = \frac{p-c}{p}$. Under sufficiently big elasticity of cost to investment expressed in condition $r_{\ln c}(f) > 1$ individual consumption decreases, otherwise it does not. The first column of the table was proved to be empty; equilibria here inexist (Table 1 also do not mention the case $r_u' = 0$, $r_{\ln c} = 1$ where equilibria are indeterminate). Existence of equilibria in the last two columns is an open question, whereas numerical examples for the middle columns are already constructed and confirm valid equilibria with all effects described by Table 1.

In the future research we plan to supplement the study of closed economy by more examples and economic interpretations. Additionally, for the version of the model with endogenous quality we plan to supplement the classification table with the behavior of more variables: $\frac{xL}{q(f)}-$ the size of the firm adjusted for quality, $pq(f)-$ the price of each variety adjusted for quality. To simplify analysis we can assume, for example, linear dependence between quality and fixed costs in the form $\tilde{f}(q) = aq$.

4. Open economy: trade model

In the previous section we have described the preliminary results for the model of closed economy. To expand this analysis to trade, now we introduce a trade model. The economy consists of two countries, "Home" (H) and "Foreign" (F), one production factor (labour) and one differentiated sector including continuum of varieties or brands. There can be a different approach: it is usual in international trade theory to assume some second sector ("agriculture") with constant returns to scale using the same labour, the trade of agricultural good going without transport costs. [4] Instead, motivated by findings in (Davis, 1998) and (Yu, 2005), we proceed

[4] This assumption allows to assume away the difficult question about the dependence of wages in the country upon the size of the country (see, for instance, classical model (Krugman, 1980) with identical producers and (Melitz, 2003) with heterogeneous producers). But, as shown in (Davis, 1998) and (Yu, 2005), the simplifying assumption of wage equalization is not so innocent and can crucially change the result. In (Davis, 1998) and (Yu, 2005) classical Krugman's model is supplemented with transport costs in agri-

with one-sector assumption, to clarify the effect of (generally non-equalized) wages on the equilibrium.

Let L be the total population of the world; $s \in [0, 1]$ is the share of population in country H (so $1-s$ is the share of population in country F). Then the population in country H equals sL, while in country F it equals $(1-s)L$. As in the case of closed economy, we assume that each consumer in each country has one unit of labour offered nonelastically. We denote wages in country H and in country F as w^H and w^F correspondingly. In what follows we normalize wages as $w^H = w$ and $w^F = 1$, without a loss of generality. Then the utility-maximization program of each consumer in country H is

$$\int_0^{N^H} u(x_i^{HH}) di + \int_0^{N^F} u(x_i^{FH}) di \to \max_{x^{HH}, x^{FH} \geq 0}$$

$$s.t.$$

$$\int_0^{N^H} p_i^{HH} x_i^{HH} di + \int_0^{N^F} p_i^{FH} x_i^{FH} di \leq w,$$

where N^H and N^F are masses of firms in country H and country F correspondingly, p_i^{HH}, x_i^{HH} denote price and individual consumption of variety i produced in country H and consumed in country H. Similarly, p_k^{FH}, x_k^{FH} are prices and consumption of the variety k produced in country F and consumed in country H. The program of a consumer in country F is analogous. [5]

Let f^H and f^F be fixed costs in countries H and F (chosen endogenously); $c^H = c(f^H)$ and $c^F = c(f^F)$ be the marginal costs in country H and country F (as before, we assume that $c'(f) < 0$). Besides, let us assume, standardly, that the trade incurs some transport costs of "iceberg type", i.e. to give to the consumer in the country F the unit of the goods, the producer in country H must produce $\tau > 1$ units of the production. Thus, taking into account different wages in country H and country F, the cost function in country H (in monetary terms) is

$$w \left(c(f^H) sL x_i^{HH} + \tau c(f^H)(1-s) L x_i^{HF} + f^H \right).$$

Similarly, cost function of the producer in in country F is

$$\tau c(f^F) sL x_j^{FH} + c(f^F)(1-s) L x_j^{FF} + f^F.$$

Then the profit-maximizing program of a producer in countries H is

$$sL(p_i^{HH}(x_i^{HH}) - wc(f^H)) x_i^{HH} + (1-s) L (p_i^{HF}(x_i^{HF}) - \tau w c(f^H)) x_i^{HF} - w f^H \longrightarrow$$

$$\longrightarrow \max_{x_i^{HH}, x_i^{HF}, f^H},$$

cultural sector, and these costs were shown to block the home market effects. However, important are not transport costs per se, but mainly the fact that these transport costs blocks the wage equalization.

[5] In country F, p_i^{HF}, x_i^{HF} mean price and consumption of variety $i$produced in country H and consumed in country F, and p_i^{FF}, x_i^{FF} relate to variety i produced in country F and consumed in country F.

and in F it is

$$sL(p_j^{FH}(x_j^{FH})-\tau c(f^F))x_j^{FH}+(1-s)L(p_j^{FF}(x_j^{FF})-c(f^F))x_j^{FF}-f^F \longrightarrow \max_{x_j^{FH},\, x_j^{FF},\, f^F}.$$

Here $p_i^{Hk}\left(x_i^{Hk}\right)$ is the inverse demand function for the commodity produced by $i-th$ firm in country H for consumption in country $k \in \{H, F\}$, and $p_j^{Fk}\left(x_j^{Fk}\right)$ is its counterpart in F.

Like in closed economy, FOC for the producer's problem, consumer's balances, free-entry (zero-profit) conditions after some calculations lead us to

Proposition 3. *Trade equilibrium* $(x^{HH},\, x^{FH},\, x^{HF},\, x^{FF},\, w,\, N^H,\, N^F,\, f^H,\, f^F)$ *is determined by equations*

$$\frac{u'(x^{HH})}{u'(x^{FH})} = \frac{wc^H}{\tau c^F} \cdot \frac{1-r_u(x^{FH})}{1-r_u(x^{HH})} \tag{8}$$

$$\frac{u'(x^{FF})}{u'(x^{HF})} = \frac{c^F}{\tau wc^H} \cdot \frac{1-r_u(x^{HF})}{1-r_u(x^{FF})} \tag{9}$$

$$N^H \cdot \frac{wc^H x^{HH}}{1-r_u(x^{HH})} + \tau N^F \cdot \frac{c^F x^{FH}}{1-r_u(x^{FH})} = w \tag{10}$$

$$\tau N^H \cdot \frac{wc^H x^{HF}}{1-r_u(x^{HF})} + N^F \cdot \frac{c^F x^{FF}}{1-r_u(x^{FF})} = 1 \tag{11}$$

$$\frac{sr_u(x^{HH})x^{HH}}{1-r_u(x^{HH})} + \frac{\tau\,(1-s)\,r_u(x^{HF})x^{HF}}{1-r_u(x^{HF})} = \frac{f^H}{c^H L} \tag{12}$$

$$\frac{\tau sr_u(x^{FH})x^{FH}}{1-r_u(x^{FH})} + \frac{(1-s)\,r_u(x^{FF})x^{FF}}{1-r_u(x^{FF})} = \frac{f^F}{c^F L} \tag{13}$$

$$c'(f^H)(sLx^{HH} + \tau(1-s)Lx^{HF}) = -1 \tag{14}$$

$$c'(f^F)(\tau sLx^{FH} + (1-s)Lx^{FF}) = -1 \tag{15}$$

$$N^H(f^H + c^H sLx^{HH} + \tau c^H(1-s)Lx^{HF}) = sL \tag{16}$$

$$N^F(f^F + \tau c^F sLx^{FH} + c^F(1-s)Lx^{FF}) = (1-s)\,L \tag{17}$$

Here equations (8) and (9) mean the optimality in consumption in countries H and F correspondingly (the ratio of marginal utilities equal the ratio of the prices, where prices are found by FOC in the producer's problem). Equations (10) and (11) are are consumer's balances in countries H and F. Equations (12) and (13) are free-entry conditions in countries H and F. Equations (14) and (15) determine optimal choice of technology in each country. Equations (16) and (17) mean balance in the labour market. Note that, as in all general equilibrium models, one of the equations can be omitted as linear dependent from others.

Now let us discuss this equations system. One can see that under absent trade costs (i.e if $\tau = 1$) there exists an equilibrium with wage and investments equalized between countries. In this case the only difference between the countries is the number of firms. This result is not very surprising since it is similar in spirit to coincidence of the integrated equilibrium and the limited-mobility equilibrium in Heckscher–Ohlin model. However, the situation changes crucially under trade cost coefficient $\tau > 1$. In this case wage and investment decisions of firms are, generally speaking, different between countries. Then an *endogenous heterogeneity of productivity/quality among firms in international trade* should appear (since f^H and f^F are different). [6]

Actually, here our aim is to study the impact of trade costs magnitude (τ) on investments per firm and total investments, and on other equilibrium variables. Is it true that when country H is bigger ($s > \frac{1}{2}$) then this country invests more per firm and has less costs per unit? What is the ratio of total investments in countries H and F? What we can say about the relation between the size of the firm, the mass of the firms and prices in these two countries? Is it true that the wage is bigger in larger country?

To answer all these questions, we plan to study comparative statics of the above equations. At the moment it seems incredible to get complete global comparative statics for these equations. At first we plan to concentrate on local comparative statics. Namely, the solution to these equations can be considered as implicit function of two parameters, s and $\frac{1}{\tau}$. Without loss of generality we can assume that couple $\left(s, \frac{1}{\tau}\right)$ belongs to the rectangle $\left[\frac{1}{2}, 1\right] \times [0, 1]$. We plan to get the answers to the questions above for the following cases: (1) $s \approx \frac{1}{2}$ and arbitrary τ - this case corresponds to the situation of the countries with similar sizes; (2) $s \approx 1$ and arbitrary τ - this case corresponds to the situation when the size of country H exceeds considerably the size of country F; (3) $\tau \approx 1$ and arbitrary $s > \frac{1}{2}$ - this case corresponds to the situation when trade barriers between countries are sufficient low; finally, (4) $\tau \approx \infty$ and arbitrary $s > \frac{1}{2}$, it corresponds to the situation of high (almost prohibitive) trade barriers. Preliminary results show that comparative statics in trade cost coefficient τ makes mass of firms N negatively correlated with investment f which is always positively connected with the size of the firm Lx. As to the market size impact, when we call elasticity relation $\mathcal{E}_{N^H/sL} > 1$ as the "Home market effect", it turns out negatively correlated with increasing individual investment, i.e., entails $\mathcal{E}_{f^H/sL} < 0$.

To better understand the equilibrium behavior, analytical results should be supplemented with computer simulations using specific functional forms of utility function and cost function.

[6] The technological heterogeneity appearing in this model is differ from the heterogeneity from Melitz's model. In Melitz's model are per se two types of heterogeneity - intra-country and inter-country. Intra-country heterogeneity is per se self-controlled by the firms. Each firm (unimportant why - for instance, due to business ability) has different degree of efficiency, and the owner of the firm can not influence by the own decision on this heterogeneity. In our setting of the model, there is no intra-country heterogeneity, but the owner of the firm fully controls the degree of the firm's efficiency. This is the main difference of our model from the Melitz's model. In other hand, in Melitz's model there is also inter-country heterogeneity that appears due to different volume of domestic markets in countries H and F and transport costs (about the role of asymmetry in Melitz's model see, e.g. (Baldwin and Forslid, 2010)). The presence of these two factors explain different degree of efficiency of boundary firm in countries H and F. This cause of inter-country heterogeneity presents also in our model.

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On Voluntariness of Nash Equilibrium

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Abstract The paper deals with pure strategy equilibria of bi-matrix games. It is argued that the set of Nash equilibria can contain voluntary as well as involuntary outcomes. Only the former are indicative of consistent expectations. In the context of repeated play with incomplete information, simulations show that involuntary equilibria tend to occur more frequently than voluntary equilibria. Consequences in econometric practice and philosophical implications are briefly hinted at.

Keywords: matrix games, expectations, pure equilibria, learning, incomplete information.

1. Introduction

In this note we discuss *voluntariness*, a psychological trait that despite its paramount importance in cognitive and behavioural sciences, seems to have been overlooked in Game Theory. It is argued that under certain conditions the very notion of Nash equilibrium together with its predictive and normative power are crucially related to the question: to what extent is the equilibrium outcome of a game what players wanted it to be? The exposition is initially framed in the simplest possible setting, one-shot bi-matrix games with known payoffs. It is assumed that rationality is exhaustively captured by players' optimizing behaviour in terms of best reply to an opponent's *expected* strategy. The mechanism by which expectations are formed is not assumed to be an ingredient of rationality, i.e. expectations may or may not be rational, consistent or inconsistent. It is shown that the set of outcomes arising from these assumptions may contain a Nash equilibrium even when expectations are inconsistent. We term *involuntary* such an equilibrium. We then remove the assumption of knowledge of the opponent's payoffs and show that the performance of a learning algorithm in presence of two pure-strategy equilibria, one voluntary one involuntary, exhibits - surprisingly - the prevalence of the latter. Our discussion dispels a common misconception present in the literature - the identification of Nash equilibrium to consistent alignment of expectations. Besides consequences in econometrics, specifically in the sort of statistical inference used to monitor agents psychologies, the question involves deeper epistemic aspects. The possibility of involuntary equilibria is akin to whether Knowledge could be assimilated to 'Justified True Belief', a thesis denied in the classical paradox of (Gettier, 1963). Furthermore, involuntary equilibria provide substantive support to the compatibilist view in the debate over Free Will vs. Determinism (Dennet, 1984).

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2. Definitions

It is well known that a Nash equilibrium in two-person games is defined as a pair of strategies $\hat{u}_1, \hat{u}_2$ satisfying

$$J_1(\hat{u}_1, \hat{u}_2) \geq J_1(u_1, \hat{u}_2) \quad \forall u_1 \in U_1 \tag{1}$$
$$J_2(\hat{u}_1, \hat{u}_2) \geq J_1(\hat{u}_1, u_2) \quad \forall u_2 \in U_2 \tag{2}$$

where J_i, U_i define the objective function (maximand) and the strategy space of player i. Definition (1-2) is equivalent to

$$\hat{u}_1 = \underset{u_1}{\operatorname{argmax}} J_1(u_1, \hat{u}_2)$$
$$\hat{u}_2 = \underset{u_2}{\operatorname{argmax}} J_2(\hat{u}_1, u_2).$$

If the best-reply operator

$$B : U_1 \times U_2 \mapsto U_1 \times U_2 \tag{3}$$
$$B = \begin{bmatrix} B_1(u_2) \\ B_2(u_1) \end{bmatrix} \tag{4}$$
$$B_1(u_2) = \underset{u_1}{\operatorname{argmax}} J_1(u_1, u_2) \tag{5}$$
$$B_2(u_1) = \underset{u_2}{\operatorname{argmax}} J_2(u_1, u_2), \tag{6}$$

is single-valued, a second characterization of Nash equilibrium can be given in terms of the fixed point[1] of B

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \hat{u} \text{ is a Nash equilibrium} \iff \quad \hat{u} = B(\hat{u}). \tag{7}$$

A third characterization, mainly focused on strategic aspects, can be given in terms of expectations. We suppose in the following B exists is continuous and single-valued over $U_1 \times U_2$.

3. Expectations

Denote by the symbol $\mathcal{E}$ the *subjective* expectation that each player has over the action of his opponent. In the case, for example, of Bayesian statistics $\mathcal{E}$ may coincide with the mathematical expectation E or with the conditional expectation $E(\cdot|\cdot)$ if posterior information is available. This of course would presuppose that players' expectations are formed on the basis of specific (and known) probability distributions over the opponent's strategy sets, perhaps in connection with externally observed events. Since any such specification would introduce a degree of arbitrariness at this stage, we stay clear of any distributional assumption. By $\mathcal{E}_j U_i$ we mean that

[1] This of course requires the operator B to be defined on a domain $\mathcal{D} \subset U_1 \times U_2$ containing the point $\hat{u}$. Unfortunately, it is frequent the case that a Nash equilibrium exists at a point where B is discontinuous, as in mixed strategy equilibria of matrix games. In other cases B may turn out to be nowhere defined *except* at the point $\hat{u}$, hence $\mathcal{D}$ may be an *a-priori* unknown domain. Even when B is defined and continuous on some domain it may not be single-valued. In game theory these are well known sources of computational difficulty.

element of U_i with respect to which player j evaluates - and implements - his own best reply, regardless of the reason why such a conviction came to materialize in his mind. We will say that player j's expectation is *fulfilled* if, after i's move is taken, it turns out $u_i = \mathcal{E}_j U_i$. Players' expectations will be said *consistent* (in two-person games) if they are both fulfilled.

Expectation Principle: if expectations are consistent, then a Nash equilibrium exists and it coincides with the expected outcome.

Proof. Being at a best-reply v_1, v_2 means

$$v_1 = \underset{u_1}{\operatorname{argmax}} J_1(u_1, \mathcal{E}_1 U_2) \tag{8}$$

$$v_2 = \underset{u_2}{\operatorname{argmax}} J_2(\mathcal{E}_2 U_1, u_2) \tag{9}$$

and if $v_1 = \mathcal{E}_2 U_1 \quad v_2 = \mathcal{E}_1 U_2$ it follows

$$J_1(\mathcal{E}_2 U_1, \mathcal{E}_1 U_2) \geq J_1(u_1, \mathcal{E}_1 U_2) \quad \forall u_1 \in U_1 \tag{10}$$

$$J_2(\mathcal{E}_2 U_1, \mathcal{E}_1 U_2) \geq J_2(\mathcal{E}_2 U_1, u_2) \quad \forall u_2 \in U_2 \tag{11}$$

hence v_1, v_2 are equilibrium strategies and coincide with the expected outcome. □

The condition is only sufficient because

$$\arg\max_{u_1} J_1(u_1, \mathcal{E}_1 U_2) = \arg\max_{u_1} J_1(u_1, v_2)$$

is implied by – but it does not imply – $v_2 = \mathcal{E}_1 U_2$ (*idem* for $\mathcal{E}_2 U_1$). In other words, u_1 may constitute a best reply to more than just one pure strategy of the opponent. If this occurs when u_1, u_2 is an equilibrium, such an equilibrium does not require player 1's expectation to be fulfilled. It may well happen that inconsistent expectations lead to a Nash equilibrium. In this case we should speak of an *involuntary* equilibrium.

To say that a given equilibrium is involuntary is better suited to the judgment of a psychologist than to the speculation of a game theorist. What *can* be established within our discipline though is whether or not the hypothesis of involuntariness can be *logically* held. Involuntariness can be ruled out in cases where *any* pair of inconsistent expectations is in dis-equilibrium. To illustrate, consider[2]

$$\begin{array}{cccc} & L & C & R \\ t & (2,3) & (4,1) & (0,1) \\ m & (1,0) & (0,4) & (5,5) \\ b & (1,3) & (2,1) & (3,0). \end{array}$$

This game has exactly two equilibria in pure strategies. These are (t, L) yielding utilities $(2, 3)$; and (m, R) yielding utilities $(5, 5)$. Suppose both players expect to be facing a *very cautious* opponent. For example, Row expects Column to play C (this is the only strategy ruling out zero utility for Column, e.g. a max-min strategy). Consequently, Row will play t, his best reply to C. Symmetrically, if Column expects Row to play b (Row's max-min strategy) her best reply is L. So the outcome associated to these expectations is (t, L). It is easy to check that this is indeed an equilibrium outcome although the expectations leading to it were *not*

[2] left numbers in brackets are utilities of Row player, right numbers of Col player.

fulfilled. They predicted (b, C), i.e. they were *inconsistent*. Hence we cannot rule out that when the equilibrium (t, L) is reached, it is reached involuntarily. To be sure, outcome (t, L) *is* a Nash equilibrium, but it does not necessarily mirror a system of self-fulfilling expectations.

Notice that this conclusion does not apply to the other equilibrium of the game. The outcome (m, R) cannot be considered an involuntary equilibrium as m is the best reply to R and to no other strategy of Column; and R is the best reply to m and to no other strategy of Row.

On the other hand consider the game dubbed *Chicken* in game theory folklore. Two drivers arrive at the same time at an intersection. Each can keep going (g) or stop (s). Both prefer (g) to (s) but only if the other stops, otherwise they prefer (s) and can be represented by the pay-off matrix

$$\begin{array}{ccc} & g & s \\ g & (0,0) & (3,1) \\ s & (1,3) & (2,2). \end{array}$$

The game has two pure strategy equilibria, (g, s) and (s, g). There are 4 possibilities for the expectations

1. *Row expects Column to stop and Column expects Row to go.* Best-reply prescribes Row to go and Column to stop, expectations are consistent, the outcome is equilibrium.
2. *Row expects Column to go and Column expects Row to stop.* Best-reply prescribes Column to go and Row to stop, expectations are consistent, the outcome is equilibrium.
3. *Both expect that the other goes.* Best-reply prescribes both to stop, expectations are inconsistent, the outcome is not equilibrium.
4. *Row expects Column to stop and Column expects Row to stop.* Best-reply prescribes both to go, expectations are inconsistent, the (disastrous) outcome is not equilibrium.

In this case involuntariness is to be ruled out on purely *algorithmic* ground: if an equilibrium is reached, it is reached voluntarily - no psychology required.

Motivated by the above, we give the following

Definition 1. A pure-strategy equilibrium of bi-matrix game is Involuntary if it contains strategies that are best-replies to more than one strategy of the opponent; otherwise is Voluntary.

4. Best-Reply Graph

A useful tool to examine the existence and the nature of pure-strategy equilibria is the Best-Reply Graph (BRG). This is a bi-partite graph

$$G(\mathbb{I}_n, \mathbb{I}_m, E_{nm}, E_{mn})$$

whose nodes are strategies and edges are best-replies. An edge $e_{ji} \in E_{nm}$ connects node $j \in \mathbb{I}_m$ to node $i \in \mathbb{I}_m$ if pure strategy i is a best-reply to pure strategy j, e.g. $i = \mathcal{B}_1(j)$, and an edge $e_{ij} \in E_{mn}$ connects node $i \in \mathbb{I}_n$ to node $j \in \mathbb{I}_n$ if pure strategy j is a best-reply to pure strategy i, e.g. $j = \mathcal{B}_2(i)$. For example the game

$$\begin{array}{cccc} & L & C & R \\ t & (6,7) & (8,4) & (0,3) \\ m & (4,5) & (3,4) & (7,6) \\ b & (5,5) & (4,0) & (5,4). \end{array}$$

has a BRG

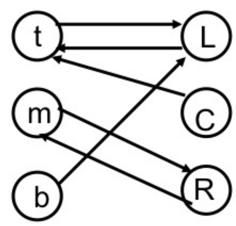

Figure1: *Best-Reply Graph*

A node with in-degree=m is a *Dominant* strategy of player 1 (similarly for player 2 with in-degree=n); nodes with in-degree= 0 are called *Never-Best-Reply* (NBR) strategies; a cycle of the BRG of length > 2 is an *Improvement Cycle*. In Fig. 1 there are no dominant strategies, b and C are NBR, and there are no improvement cycles. Moreover,

i. each node of a BRG has out-degree> 0
ii. nodes of Nash equilibria belong to cycles of length$= 2$
iii. nodes of *strict* equilibria have out-degree$= 1$
iv. an equilibrium is involuntary if at least one of its nodes has in-degree> 1
v. every potential game has at least one pure-strategy equilibrium
vi. potential games have no improvement cycles.

i. follows from existence of B; ii., iii. and iv. hold by definition; v. and vi. are well known (e.g. Cor 2.2 and 2.3 in (Monderer and Shapley, 1996)). The game in Fig.1 has one voluntary equilibrium (m, R) and one involuntary equilibrium (t, L).

Remark 1. Notice that involuntary equilibria are all but rare in matrix games. For instance, all equilibria in dominant strategies are involuntary since by definition a domininant strategy is a best-reply to more than one (indeed to any) of the opponent's strategies. Perhaps the epitomy of equilibria in dominant strategies is the Prisoner's Dilemma[3]

$$\begin{array}{ccc} & c & d \\ c & (1,1) & (3,0) \\ d & (0,3) & (2,2) \end{array}$$

[3] pay-offs are years of prisonment.

where Deny (d) dominates Confess (c) for both players. If both players expect (c) from the opponent, their best replies lead to (d, d) with inconsistent expectations, i.e. (d, d) is an involuntary equilibrium.

Furthermore

Proposition 1. *Every potential game with $n \neq m$ has at least one Involuntary Equilibrium*

Proof. If Row-strategies are $r_1 \dots r_n$ and Col-strategies are $c_1 \dots c_m$ (take $n < m$ wlog) re-number nodes such that the equilibria of the potential game are $(r_i, c_i), i \in \mathcal{P}$ and assume they are all voluntary. If $|\mathcal{P}| = n$, then there is an ougoing edge

$$c_h \mapsto r_k \quad \text{ with } h \notin \mathcal{P}, k \in \mathcal{P}$$

showing that r_k has indegree > 1 hence (r_k, c_k) is involuntary. If $|\mathcal{P}| < n$, use outgoing edges to re-order nodes with index $k \notin \mathcal{P}$

$$r_{k_1} \mapsto c_{k_1} \mapsto r_{k_2} \mapsto c_{k_2} \dots \mapsto r_n \mapsto c_n$$

Since c_n must have one outgoing edge

$$c_n \mapsto r_i \quad \text{ for some } i \in \{1 \dots n\}$$

But $i = n$ for otherwise there would be an improvement cycle and the game would not be potential. Thus (r_n, c_n) is a 2-cycle containing node r_n with in-degree> 1, i.e. an Involuntary Equilibrium. □

This result confirms the set of cases in which matrix games exhibit involuntary equilibria is of non-negligible importance.

5. Incomplete information

Notice that *involuntariness* is entirely based on *best-reply* and not at all on *expectations* nor on the mechanism by which they are formed. In the first example of Sec. 3 we assumed, for the sake of the argument, each player expected the opponent to be very cautious. Given the structure of the payoff matrices, that seemed a plausible argument: individual rationality may well motivate max-min strategies, for example when common knowledge cannot be invoked. However involuntary equilibria may arise with completely irrational expectations. For example, in the game of Fig. 1 max-min rationality would suggest the row-player to play b rather than t. In fact, expectations not even require to be formed on the basis of *known* opponents payoffs. Assume players only know their *own* payoffs - not their opponent's. In the frame of repeated games with simultaneous moves, let each player know only *own* current and *own* past performance. Assume players adopt, among the many proposed, a so called linear-reward learning scheme (Thathachar and Sastry, 2004). It works like this. A pure strategy s of player 1 is sampled randomly at round t, according to a probability vector $p(t) \in S^n$ (unit simplex of $\mathbb{R}^n$) which updates as

$$p_i(t+1) = \begin{cases} p_i(t) + \eta P(t)(1 - p_i(t)) & \text{if } i = s \\ p_i(t) - \eta P(t) p_i(t) & \text{if } i \neq s \end{cases} \tag{12}$$

where $P(t)$ is the outcome payoff of player 1 normalized to the unit interval, and η a learning rate chosen in the same interval. Similarly, player 2 selects pure strategy r sampled with probability $q(t) \in S^m$ with

$$q_i(t+1) = \begin{cases} q_i(t) + \eta Q(t)(1 - q_i(t)) & \text{if } i = r \\ q_i(t) - \eta Q(t) q_i(t) & \text{if } i \neq r. \end{cases} \tag{13}$$

It is known (Sastry et al., 1994) that, for small η, each pure-strategy strict Nash equilibrium is a locally asymptotically stable point of (12,13) and any point which is not a Nash equilibrium is unstable.

Given the results of the previous section it is interesting to use the above learning scheme to assess the relative frequency of occurrence of an involuntary equilibrium in the class of potential games possessing one voluntary and one involuntary equilibrium. The first example of Sec. 3 belongs to this class and the above learning scheme typically produces results of the kind shown below. Initial probabilities p, q are all set to 1/3 and $\eta = 0.1$. Fig. 2 shows convergence of p to [0 1 0] (top) and of q to [0 0 1] (bottom). This is the voluntary equilibrium (m, R) yielding payoffs $(5, 5)$.

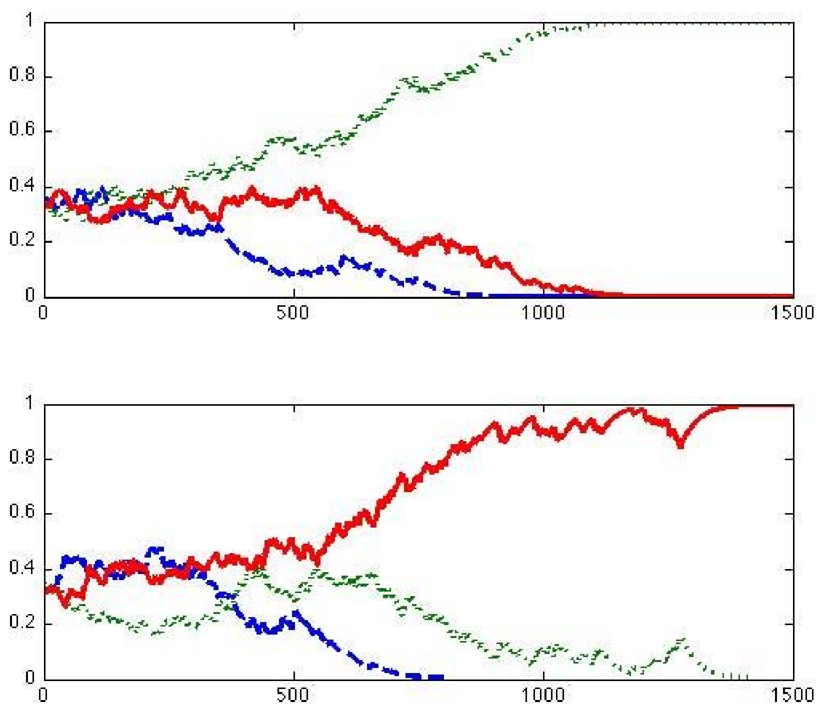

Figure2: *Convergence of* p *(top) and* q *(bottom) to voluntary equilibrium*

Fig. 3 shows convergence of p to [1 0 0] (top) and of q to [1 0 0] (bottom). This is the involuntary equilibrium (t, L) yielding payoffs $(2, 3)$. Convergence to one equilibrium or the other is regulated by chance, given the random sampling that takes place at each round. The question of interest is long-run behaviour: which of the two outcomes prevails in a reiterated simulation cycle.

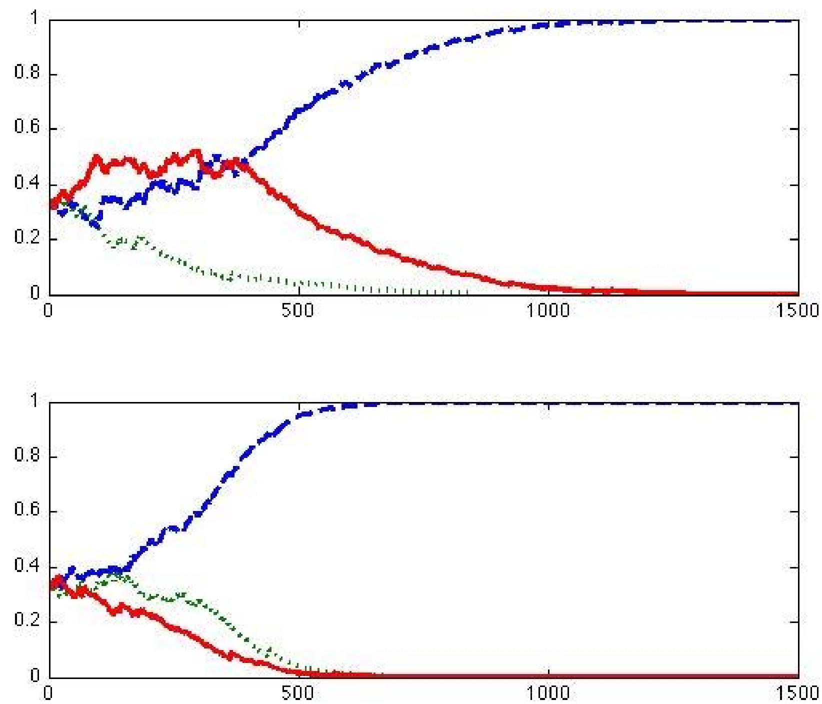

Figure3: *Convergence of p, q to involuntary equilibrium*

In Fig. 4 (bottom) the simulations shown in Figs. 1,2 have been replicated 2000 times. The dashed line represents the number of times an involuntary equilibrium has been reached, the solid line the number of times a voluntary equilibrium has been reached. The top graph of Fig. 4 (top) is a zoom on the first 20 rounds. It is apparent that in the long-run the involuntary equilibrium prevails.

6. Discussion and conclusion

The idea of relating equilibrium to self-fulfilling expectations pervades Economics before and after the onset of the rational expectations paradigm. Expectations determine behaviour. The standard problem is to deduce equilibrium price and quantities from consumer and producer's preferences. However inverse questions can be – and usually are – posed in Econometrics, including for example guessing consumers preferences from market-clearing prices; assessing investors risk-preferences from bond yields; dubbing stockmarket attitudes bullish or bearish, etc. Drawing positive conclusions on the nature of expectations from revealed behaviour means reversing the expectations-behaviour link somewhat arbitrarily through statistical inference. That self-fulfilling expectations lead to equilibrium is a truism following

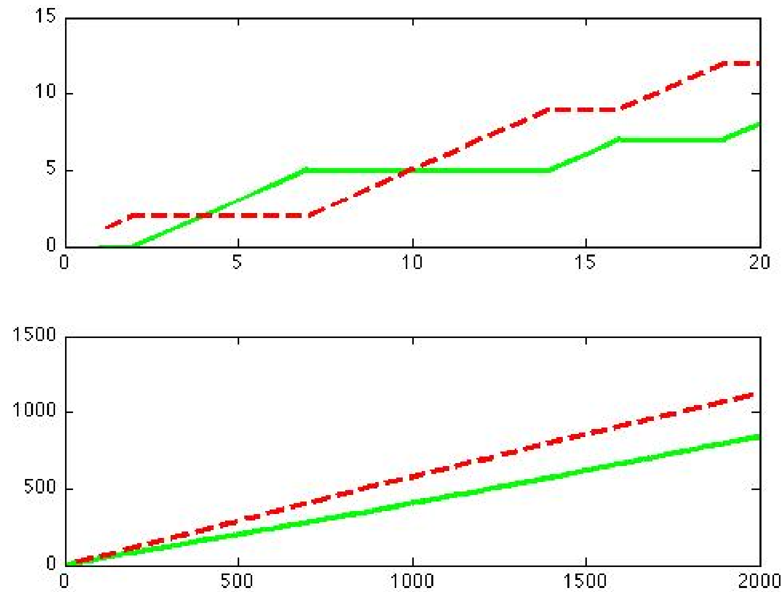

Figure4: *Solid: number of times VOL is reached. Dash: number of times INVOL is reached.*

straight from the definition of Nash (1,2). Revealing agents' expectation from observed equilibrium is a far more ambitious and definitely non-trivial task. Indeed, this task is not always viable and often doomed to fail. This is so, it is argued in this note, when equilibria are involuntary.

Consistent alignment of beliefs is not a new idea. The term *rationalizable* has been used to describe strategies a player can defend (i.e. rationalise) on the basis of beliefs about the beliefs of the opponent that are not inconsistent with the game's data (Bernheim, 1984), (Pearce, 1984). More precisely, rationalizable equilibria are outcomes of a game in which dominated strategies are iteratively eliminated (see also the notion of sophisticated equilibria in (Moulin, 1986)). Although rationalizable outcomes need not be Nash equilibria, every Nash equilibrium is a rationalizable outcome. But this is not to say that Nash equilibrium is sustained by consistent beliefs, as we have just shown. Despite this, some confusion seems to be present in the literature. For example, in (Hargreaves Heap and Varoufakis, 1995Sec 1.2.1) we read

> Nash strategies are the only rationalizable ones which, if implemented, confirm the expectations on which they were based. This is why they are often referred to as self-confirming strategies or why it can be said that this equilibrium concept *requires* that players' beliefs are consistently aligned.

The implicit root of our question lies deep in epistemology, namely in the analysis of Knowledge and its controversial identification with Justified Belief (Dennet, 1989). Involuntary equilibria are akin to the Gettier case (Gettier, 1963) rephrased here as

> Smith has applied for a job but, from rumors, has a justified belief that "Jones will get the job". Also he knows "Jones has 10 coins in his pocket", as he counted the coins in Jones's pocket ten minutes ago. Smith therefore (justifiably) concludes that "the man who will get the job has 10 coins in his pocket". In fact, Jones does not get the job. Instead, Smith does. However, as it happens, Smith (unknowingly and by sheer chance) also had 10 coins in his pocket. So his belief that "the man who will get the job has 10 coins in his pocket" was justified and true. But it does not appear to be knowledge.

The similarity is only partial. Whereas justified beliefs are necessary but not sufficient to Knowledge in the case of Gettier, they are sufficient but not necessary to reach an involuntary equilibrium. Indeed they not even need to be *justified* in this case.

Now if free-will originates from beliefs and beliefs are irrelevant to outcomes, then free-will is irrelevant to outcomes. Thus involuntary equilibria are an instance of determinism. Game theory, to the extent it can exhibit both voluntary and involuntary equilibria, offers a powerful paradigm in favour of compatibilism in the debate (Dennet, 1984) over Free-will vs. Determinism.

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Model Differential Game with Two Pursuers and One Evader*

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Abstract An antagonistic differential game is considered where motion occurs in a straight line. Deviations between the first and second pursuers and the evader are computed at the instants T_1 and T_2, respectively. The pursuers act in coordination. Their aim is to minimize the resultant miss, which is equal to the minimum of the deviations happened at the instants T_1 and T_2. Numerical study of value function level sets (Lebesgue sets) for qualitatively different cases is given.

Keywords: pursuit-evasion differential game, linear dynamics, value function.

1. Introduction and Problem Formulation

In the paper (Ganebny et al., 2011), we have started a systematic study of the following differential game.

Three inertial objects moves in the straight line. The dynamics descriptions for pursuers P_1 and P_2 are

$$\begin{array}{ll} \ddot{z}_{P_1} = a_{P_1}, & \ddot{z}_{P_2} = a_{P_2}, \\ \dot{a}_{P_1} = (u_1 - a_{P_1})/l_{P_1}, & \dot{a}_{P_2} = (u_2 - a_{P_2})/l_{P_2}, \\ |u_1| \le \mu_1, & |u_2| \le \mu_2, \\ a_{P_1}(t_0) = 0, & a_{P_2}(t_0) = 0. \end{array} \tag{1}$$

Here, z_{P_1} and z_{P_2} are the geometric coordinates of the pursuers, a_{P_1} and a_{P_2} are their accelerations generated by the controls u_1 and u_2. The time constants l_{P_1} and l_{P_2} define how fast the controls affect the systems.

The dynamics of the evader E is similar:

$$\begin{array}{l} \ddot{z}_E = a_E, \;\; \dot{a}_E = (v - a_E)/l_E, \\ |v| \le \nu, \quad\; a_E(t_0) = 0. \end{array} \tag{2}$$

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Let us fix some instants T_1 and T_2. At the instant T_1, the miss of the first pursuer with the respect to the evader is computed, and at the instant T_2, the miss of the second one is computed:

$$r_{P_1,E}(T_1) = |z_E(T_1) - z_{P_1,E}(T_1)|, \quad r_{P_2,E}(T_2) = |z_E(T_2) - z_{P_2,E}(T_2)|. \tag{3}$$

Assume that the pursuers act in coordination. This means that we can join them into one player P (which will be called the *first player*). This player governs the vector control $u = (u_1, u_2)$. The evader is counted as the *second player*. The result miss is the following value:

$$\varphi = \min\{r_{P_1,E}(T_1),\ r_{P_2,E}(T_2)\}. \tag{4}$$

At any instant t, all players know exact values of all state coordinates z_{P_1}, $\dot z_{P_1}$, a_{P_1}, z_{P_2}, $\dot z_{P_2}$, a_{P_2}, z_E, $\dot z_E$, a_E. The first player choosing its feedback control minimizes the miss φ, the second one maximizes it.

Relations (1)–(4) define a standard antagonistic differential game with linear dynamics. One needs to construct the value function of this game.

The main difficulty of studying game (1)–(4) is not that $T_1 \neq T_2$, generally speaking. Game (1)–(4) is difficult and interesting due to non-convexity of the payoff function even when $T_1 = T_2$. Emphasize that we do not apply any limiting conditions of "uniformity" of the objects under consideration. Conditions of this type are usual for problems of group pursuit; see, for example, following books (Petrosjan, 1977), (Rikhsiev, 1989), (Grigorenko, 1990), (Chikrii, 1997), (Blagodatskih and Petrov, 2009).

In the paper (Ganebny et al., 2011), we analyze solutions of game (1)–(4) for two extreme cases: 1) both pursuers P_1 and P_2 are dynamically stronger than the evader E; 2) both pursuers are dynamically weaker.

This paper deals with studying level sets of the value function for intermediate cases of the game.

2. Passage to Two-Dimensional Differential Game

Let us apply to game (1)–(4) the standard passage to an equivalent differential game of the order 2 on the phase variable.

At first, let us pass to relative geometric coordinates

$$y_1 = z_E - z_{P_1}, \quad y_2 = z_E - z_{P_2} \tag{5}$$

in dynamics (1), (2) and payoff function (4). After this, we have the following notations:

$$\begin{aligned}
&\ddot y_1 = a_E - a_{P_1}, && \ddot y_2 = a_E - a_{P_2},\\
&\dot a_{P_1} = (u_1 - a_{P_1})/l_{P_1}, && \dot a_{P_2} = (u_2 - a_{P_2})/l_{P_2},\\
&\dot a_E = (v - a_E)/l_{P_1}, && |u_2| \le \mu_2,\\
&|u_1| \le \mu_1,\ |v| \le \nu, && \varphi = \min\{|y_1(T_1)|,\ |y_2(T_2)|\}.
\end{aligned} \tag{6}$$

State variables of system (6) are y_1, $\dot y_1$, a_{P_1}, y_2, $\dot y_2$, a_{P_2}, a_E; u_1 and u_2 are controls of the first player; v is the control of the second one. The payoff function φ depends on the coordinate y_1 at the instant T_1 and on the coordinate y_2 at the instant T_2.

A standard approach to study linear differential games with fixed terminal instant and payoff function depending on some state coordinates at the terminal instant is to pass to new state coordinates (see (Krasovskii and Subbotin, 1974), (Krasovskii and Subbotin, 1988)) that can be treated as values of the target coordinates forecasted to the terminal instant under zero controls. Often, these coordinates are called the *zero effort miss coordinates* (Shinar and Gutman, 1980), (Shima and Shinar, 2002), (Shinar and Shima, 2002). In our case, we have two instants T_1 and T_2, but coordinates computed at these instants are independent; namely, at the instant T_1, we should take into account $y_1(T_1)$ only, and at the instant T_2, we use the value $y_2(T_2)$. This fact allows us to use the mentioned approach when solving the differential game (6). With that, we pass to new state coordinates x_1 and x_2 where $x_1(t)$ is the value of y_1 forecasted to the instant T_1 and $x_2(t)$ is the value of y_2 forecasted to the instant T_2.

The forecasted values are computed by formula

$$x_i = y_i + \dot{y}_i\tau_i - a_{P_i} l_{P_i}^2 h(\tau_i/l_{P_i}) + a_E l_E^2 h(\tau_i/l_E), \quad i = 1, 2. \tag{7}$$

Here, x_i, y_i, $\dot{y}_i$, a_{P_i}, and a_E depend on t; $\tau_i = T_i - t \geq 0$. Function h is described by the relation

$$h(\alpha) = e^{-\alpha} + \alpha - 1.$$

Emphasize that the values τ_1 and τ_2 are connected to each other by the relation $\tau_1 - \tau_2 = \text{const} = T_1 - T_2$. It is very important that $x_i(T_i) = y_i(T_i)$. Let $X(t, z)$ be a two-dimensional vector composed of the variables x_1, x_2 defined by formulae (5) and (7).

The dynamics in the new coordinates x_1, x_2 is the following (Le Ménec, 2011):

$$\begin{aligned}
&\dot{x}_1 = -l_{P_1} h(\tau_1/l_{P_1}) u_1 + l_E h(\tau_1/l_E) v,\\
&\dot{x}_2 = -l_{P_2} h(\tau_2/l_{P_2}) u_2 + l_E h(\tau_2/l_E) v,\\
&|u_1| \leq \mu_1,\ |u_2| \leq \mu_2,\ |v| \leq \nu,\\
&\varphi\big(x_1(T_1), x_2(T_2)\big) = \min\{|x_1(T_1)|,\ |x_2(T_2)|\}.
\end{aligned} \tag{8}$$

The first player governs the controls u_1, u_2 and minimizes the payoff φ; the second one has the control v and maximizes φ. Using system 8, we assume that if $T_1 > T_2$ and $t_0 \in (T_2, T_1]$ then $\varphi = \big|x_1(T_1)\big|$; if $T_2 > T_1$ and $t_0 \in (T_1, T_2]$ then $\varphi = \big|x_2(T_2)\big|$.

Note that the control u_1 (u_2) affects only the horizontal (vertical) component $\dot{x}_1$ ($\dot{x}_2$) of the velocity vector $\dot{x} = (\dot{x}_1, \dot{x}_2)^{\mathrm{T}}$. When $T_1 = T_2$, the second summand in dynamics (8) is the same for $\dot{x}_1$ and $\dot{x}_2$.

Let $x = (x_1, x_2)^{\mathrm{T}}$ and $V(t, x)$ be the value of the value function of game (8) at the position (t, x). From general results of the theory of differential games, it follows that

$$\mathcal{V}(t, z) = V\big(t, X(t, z)\big). \tag{9}$$

Relation (9) allows to compute the value function of the original game (1)–(4) using the value function for game (8).

For any $c \geq 0$, a level set (a Lebesgue set)

$$W_c = \big\{(t, x) : V(t, x) \leq c\big\}$$

of the value function in game (8) can be treated as the solvability set for the considered game with the result not greater than c, that is, for a differential game with dynamics (8) and the terminal set

$$M_c = \big\{(t,x) : t = T_1,\ |x_1| \le c\big\} \ \cup\ \big\{(t,x) :\ t = T_2,\ |x_2| \le c\big\}.$$

When $c = 0$, one has the situation of the exact capture. The exact capture means equality to zero of, at least, one of $x_1(T_1)$ and $x_2(T_2)$.

Let

$$W_c(t) = \{x : (t,x) \in W_c\}$$

be the time section (t-section) of the set W_c at the instant t. Similarly, let $M_c(t)$ for $t = T_1$ and $t = T_2$ be the t-section of the set M_c at the instant t.

Comparing dynamics capabilities of each of pursuers P_1 and P_2 and the evader E, one can introduce the parameters (Shinar and Shima, 2002), (Le Ménec, 2011)

$$\eta_i = \mu_i/\nu, \quad \varepsilon_i = l_E/l_{P_i}, \quad i = 1,2.$$

They define the shape of the solvability sets in the individual games P_1 against E and P_2 against E.

Namely, depending on values of η_i and $\eta_i\varepsilon_i$ (which are not equal to 1 simultaneously), there are 4 cases (Shinar and Shima, 2002) of the solvability set evolution (see Fig. 1):

– expansion in the backward time (a strong pursuer);
– contraction in the backward time (a weak pursuer);
– expansion until some backward time instant and further contraction;
– contraction until some backward time instant and further expansion (if the solvability set still has not broken).

In this paper, we study level sets of the value function for the following cases: 3) one of the pursuers is stronger than the evader, and the second one is weaker;

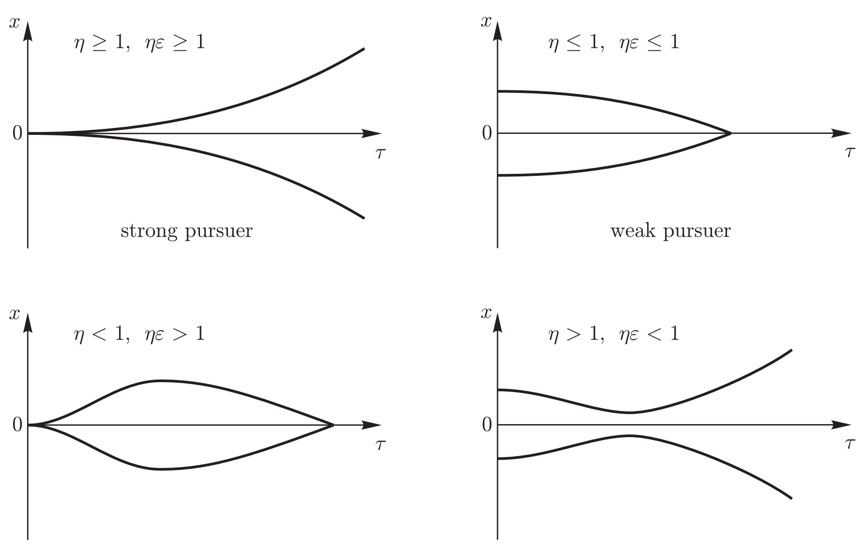

Figure1: Variants of the solvability set evolution in an individual game

4) dynamic capabilities of the pursuers P_1 and P_2 are equal; corresponding individual solvability sets contract at the beginning of the backward time and expand further.
5) solvability sets in the game $P_1 - E$ are as in Fig. 1 in bottom-left, and solvability sets in the game $P_2 - E$ are as in Fig. 1 in bottom-right.

Up to now, many algorithms have been suggested for numeric solution of differential games of quite general type (see, for example, (Cardaliaguet et al., 1999), (Mitchell, 2002), (Taras'ev et al., 2006), (Cristiani and Falcone, 2009)). We study problem (8), which is of the second order in the phase variable and can be rewritten as

$$\begin{aligned} &\dot{x} = \mathcal{D}_1(t)u_1 + \mathcal{D}_2(t)u_2 + \mathcal{E}(t)v, \\ &|u_1| \le \mu_1,\ |u_2| \le \mu_2,\ |v| \le \nu. \end{aligned} \tag{10}$$

Here, $x = (x_1, x_2)^{\mathrm{T}}$; vectors $\mathcal{D}_1(t)$, $\mathcal{D}_2(t)$, and $\mathcal{E}(t)$ are defined as

$$\begin{aligned} &\mathcal{D}_1(t) = \big(-l_{P_1} h((T_1 - t)/l_{P_1})^{\mathrm{T}},\, 0\big), \qquad \mathcal{D}_2(t) = \big(0,\, -l_{P_2} h((T_2 - t)/l_{P_2})\big)^{\mathrm{T}}, \\ &\mathcal{E}(t) = \big(l_E h((T_1 - t)/l_E),\, l_E h((T_2 - t)/l_E)\big)^{\mathrm{T}}. \end{aligned}$$

The control of the first player has two independent components u_1 and u_2. The vector $\mathcal{D}_1(t)$ ($\mathcal{D}_2(t)$) is directed along the horizontal (vertical) axis. The second player's control v is scalar.

Due to specifity of our problem, we use special methods for constructing level sets of the value function. This allows us to make very fast computations of variants of the game.

3. Maximal Stable Bridge: Control with Discrimination of Opponent. The Main Idea of Backward Numerical Construction

A level set W_c of the value function V is a maximal stable bridge (MSB) (see (Krasovskii and Subbotin, 1974), (Krasovskii and Subbotin, 1988)), which breaks on the terminal set M_c.

Let $T_1 = T_2$. Denote $T_f = T_1$. Using the concept of MSB, we can say that W_c is the set maximal by inclusion in the space $(t \le T_f, x)$ such that $W_c(T_f) = M_c(T_f)$ and the *stability* property holds: for any position $(t_*, x_*) \in W_c(t_*)$, $t_* < T_f$, any instant $t^* > t_*$, $t^* \le T_f$, any constant control v of the second player, which obeys the constraint $|v| \le \nu$, there is a measurable control $t \to \big(u_1(t), u_2(t)\big)^{\mathrm{T}}$ of the first player, $t \in [t_*, t^*)$, $|u_1(t)| \le \mu_1$, $|u_2(t)| \le \mu_2$, guiding system (8) from the state x_* to the set $W_c(t^*)$ at the instant t^*.

The stability property assumes discrimination of the second player by the first one: the choice of the first player's control in the interval $[t_*, t^*)$ is made after the second player announces his control in this interval.

It is known (Krasovskii and Subbotin, 1974), (Krasovskii and Subbotin, 1988) that any MSB is close. The set

$$W_c^{(2)}(t) = \mathrm{cl}\,\big(R^2 \setminus W_c(t)\big)$$

(here, the symbol cl denotes the operation of closure) is the time section of MSB $W_c^{(2)}$ for the second player at the instant t. The bridge terminates at the instant T_f on the set $M_c^{(2)}(T_f) = \mathrm{cl}\big(R^2 \setminus M_c(T_f)\big)$. If the initial position of system (8) is in $W_c^{(2)}$

and if the first player is discriminated by the second one, then the second player is able to guide the motion of the system to the set $M_c^{(2)}(T_f)$ at the instant T_f. Thus, $\partial W_c = \partial W_c^{(2)}$. It is proved that for any initial position $(t_0, x_0) \in \partial W_c$, the value c is the best guaranteed result for the first (second) player in the class of feedback controls.

Presence of an idealized element (the discrimination of the opponent) allowed to create effective numerical methods for backward construction of MSBs (see, for example, (Ushakov, 1998)). Linearity of the dynamics and two-dimensionality of the state variable simplify the algorithms sufficiently.

The algorithm, which is suggested by the authors for constructing the approximating sets $\widetilde{W}_c(t)$, uses a time grid in the interval $[0, T_f]$: $t_N = T_f$, t_{N-1}, t_{n-2}, $\ldots$. For any instant t_k from the taken grid, the set $\widetilde{W}_c(t_k)$ is built on the basis of the previous set $\widetilde{W}_c(t_{k+1})$ and a dynamics obtained from (8) by fixing its value at the instant t_{k+1}. So, dynamics (8), which varies in the interval $(t_k, t_{k+1}]$, is changed by a dynamics with simple motions (Isaacs, 1965). The set $\widetilde{W}_c(t_k)$ is regarded as a collection of all positions at the instant t_k where from the first player guarantees guiding the system to the set $\widetilde{W}_c(t_{k+1})$ under "frozen" dynamics (8) and discrimination of the second player. The corresponding formula has the form

$$\widetilde{W}_c(t_k) = \big(\widetilde{W}_c(t_{k+1}) - (t_{k+1} - t_k)\mathcal{D}(t_{k+1}) \cdot P\big) \overset{*}{-} (t_{k+1} - t_k)\mathcal{E}(t_{k+1}) \cdot Q. \tag{11}$$

Here, $\mathcal{D}(t_{k+1})$ is a matrix composed of columns $\mathcal{D}_1(t_{k+1})$ and $\mathcal{D}_2(t_{k+1})$ of system (10); the sets P and Q are

$$P = \big\{(u_1, u_2) : |u_1| \le \mu_1,\ |u_2| \le \mu_2\big\}, \quad Q = \{v : |v| \le \nu\}.$$

The symbol $\overset{*}{-}$ denotes the geometric difference (Minkowski difference) of two sets:

$$\mathcal{A} \overset{*}{-} \mathcal{B} = \bigcap_{b \in \mathcal{B}} (\mathcal{A} - b).$$

The boundary condition for the recursive computations (11) is assumed to be $\widetilde{W}_c(t_N) = M_c(T_f)$.

Due to symmetry of dynamics (8) and the set $W_c(T_f)$ with respect to the origin, one gets that for any $t \le T_f$ the time section $W_c(t)$ is symmetric also.

If $T_1 \ne T_2$, then there is no appreciable complication in constructing MSBs for the problem considered in comparison with the case $T_1 = T_2$. Indeed, let $T_1 > T_2$. Then in the interval $(T_2, T_1]$ in (8), we take into account only the dynamics of the variable x_1 when building the bridge W_c backwardly from the instant T_1. With that, the terminal set at the instant T_1 is taken as $M_c(T_1) = \{(x_1, x_2) : |x_1| \le c\}$. When the constructions are made up to the instant T_2, we add the set $M_c(T_2)$, that is, we take

$$W_c(T_2) = W_c(T_2 + 0) \bigcup \big\{(x_1, x_2) : |x_2| \le c\big\},$$

and further constructions are made on the basis of this set.

So, our tool for finding a level set of the value function in game (8) corresponding to a number c is the backward procedure for constructing a MSB with the terminal set M_c.

The solvability set with the index equal to c in the individual game $P1$–E ($P2$–E) is the maximal stable bridge built in the coordinates t, x_1 (t, x_2) and terminating at the instant T_1 (T_2) on the set $|x_1| \le c$ ($|x_2| \le c$). Its t-section, if it is non-empty, is a segment in the axis x_1 (x_2) symmetric with respect to the origin. In the plane x_1, x_2, this segment corresponds to a vertical (horizontal) strip of the same width near the axis x_2 (x_1). It is evident that when $t \le T_1$ ($t \le T_2$) such a strip is contained in the section $W_c(t)$ of MSB W_c of game (8) with the terminal set M_c.

4. One Strong and One Weak Pursuers

Let us take the following parameters of the game:

$$\mu_1 = 2, \quad \mu_2 = 1, \quad \nu = 1, \quad l_{P_1} = 1/2, \quad l_{P_2} = 1/0.3, \quad l_E = 1.$$

In this case, the evader is more maneuverable than the second pursuer, and an exact capture by this pursuer is unavailable. Assume $T_1 = 5$, $T_2 = 7$.

In Fig. 2, there are sections of MSB $W_{5.0}$ (that is, $c = 5.0$) for 6 instants: $t = 7.0,\ 5.0,\ 2.5,\ 1.4,\ 1.0,\ 0.0$. The horizontal part of its time section $W_{5.0}(\tau)$ decreases with growth of τ, and breaks further. The vertical part grows. After breaking the individual stable bridge of the second pursuer (and respective collapse of the horizontal part of the cross), there is the vertical strip only with two additional parts determined by the joint actions of both pursuers.

The set W_c in the space t, x_1, x_2 for $c = 5.0$ is shown in Fig. 3 from two points of view. During evolution of the sections $W_{5.0}(t)$ in t, they change their structure at some instants. These places are marked by drops in the constructed surface of the set.

Time sections $\{W_c(t)\}$ are given in Fig. 4 at the instant $t = 1$ ($\tau_1 = 4$, $\tau_2 = 6$), and at the instant $t = 4$ ($\tau_1 = 1$, $\tau_2 = 3$). There are 9 MSBs for c from 12 to 20 with the step 1.

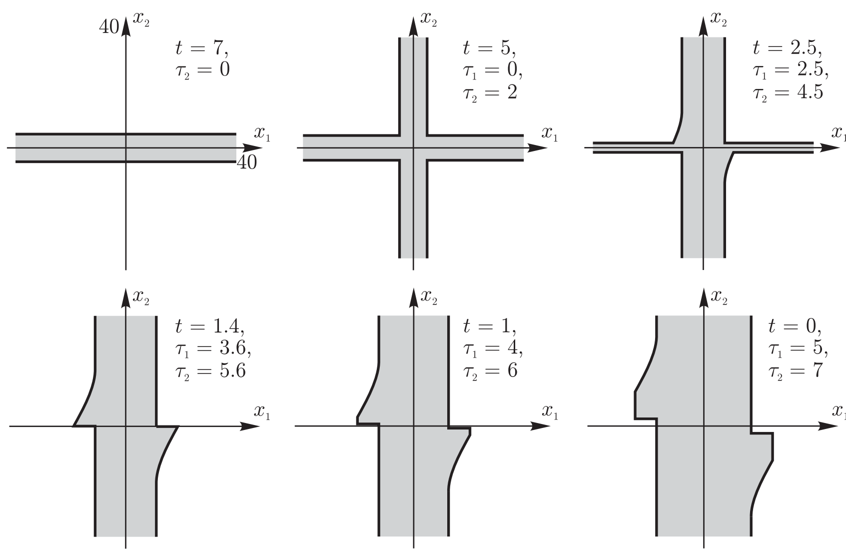

Figure2: One strong and one weak pursuers, different termination instants: time sections of the maximal stable bridge $W_{5.0}$

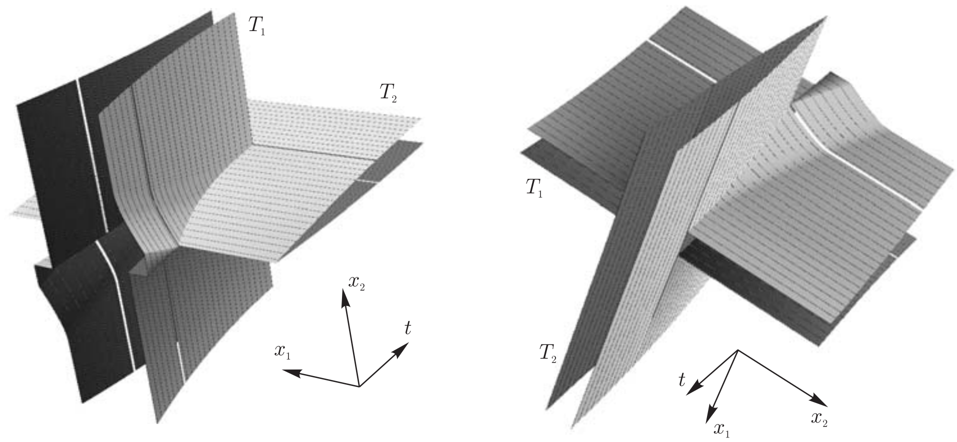

Figure3: One strong and one weak pursuers, different termination instants: two three-dimensional views of the maximal stable bridge $W_{5.0}$

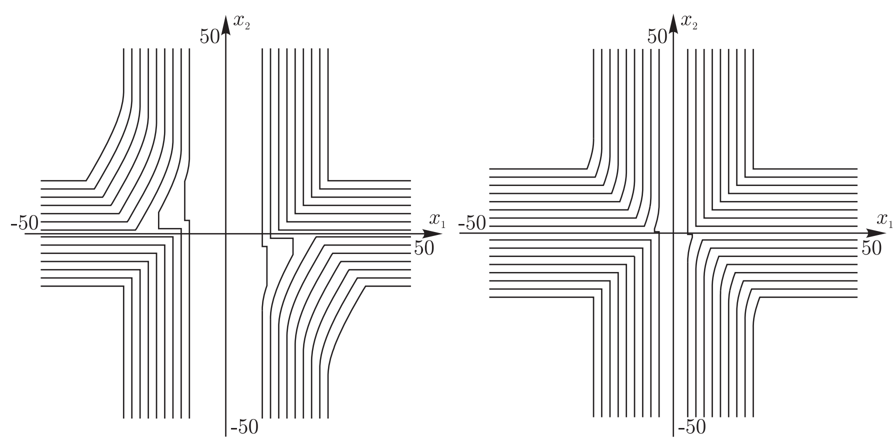

Figure4: One strong and one weak pursuers, different termination instants: time sections of MSBs at $t = 1$ (*at the left*) and at $t = 4$ (*at the right*)

5. Varying Advantage of Pursuers

5.1. Variant 1

Let us pass to the case of varying advantage of pursuers. Consider a variant when both pursuers P_1 and P_2 are equal, with that at the beginning of the backward time, the bridges in the individual games contract and further expand. Choose the game parameters in such a way that for some c the section $W_c(t)$ of MSB W_c with decreasing of t disjoins into two parts, which join back with further decreasing of t.

Parameters of the game are taken as follows:

$$\mu_1 = \mu_2 = 1.1, \quad \nu = 1, \quad l_{P_1} = l_{P_2} = 1/0.6, \quad l_E = 1.$$

Termination instants are equal: $T_1 = T_2 = 20$.

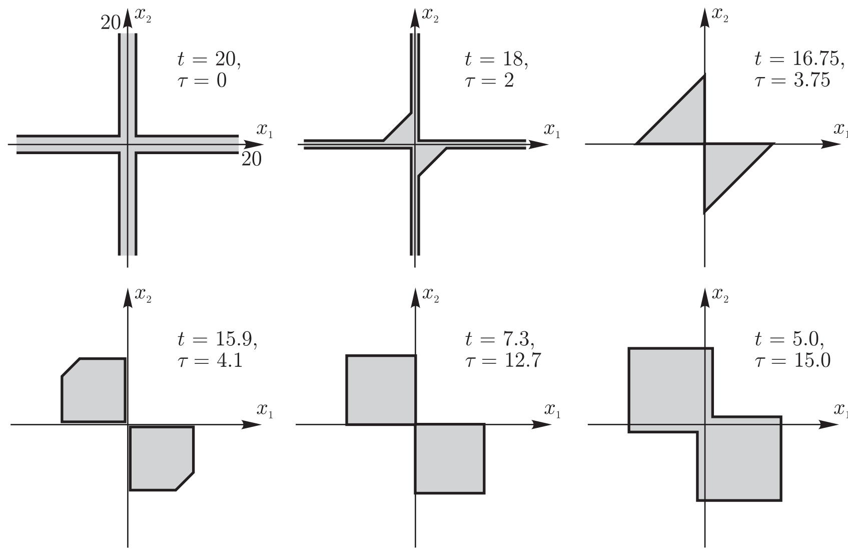

Figure5: Varying advantage of the pursuers, variant 1: time sections of the maximal stable bridge $W_{0.526}$

In Fig. 5, the time sections of MSB $W_{0.526}$ are shown for 6 instants: $t = 20.0$, 18.0, 16.75, 15.9, 7.5, 5.0. At the termination instant, the terminal set is taken as a cross (the upper-left subfigure).

At the beginning of backward time, the widths of both vertical and horizontal strips of the “cross” decreases, and two straight-linear additional triangles of joint capture zone appear (the upper-middle subfigure). Then at some instant, both strips collapse, and only the triangles constitute the time section of the bridge (the upper-right subfigure). Further, the triangles continue to contract, so they become two pentagons separated by an empty space near the origin (the lower-left subfigure). Transformation to pentagons can be explained in the following way: the first player using its controls expands the triangles vertically and horizontally, and the second player contracts them in diagonal direction. So, vertical and horizontal edges appear, but the diagonal part becomes shorter. Also, in general, size of each figure decreases slowly.

Due to action of the second player, the diagonal disappears and the pentagons convert to squares at some instant (this is not shown in Fig. 5). After that, the pursuers have advantage, and total contraction is changed by growth: the squares start to enlarge. After some time passes, the squares touch each other at the origin due to the growth (the lower-middle subfigure). Since the enlargement continues, their sizes grow, and the squares start to overlap forming one “eight-like” shape (the lower-right subfigure).

Three-dimensional views of MSBs W_c corresponding to $c = 0.526$ and $c = 3.684$ are shown in Fig. 6. Backward construction in this figure are made up to the instant $t = 5.0$ ($\tau = 15.0$).

Fig. 7 shows time sections $\{W_c(t)\}$ of a collection of MSBs for the instants $t = 12.5$ and $t = 16.0$. There are 12 MSBs for c from 0.5 to 6.0 with the step 0.5.

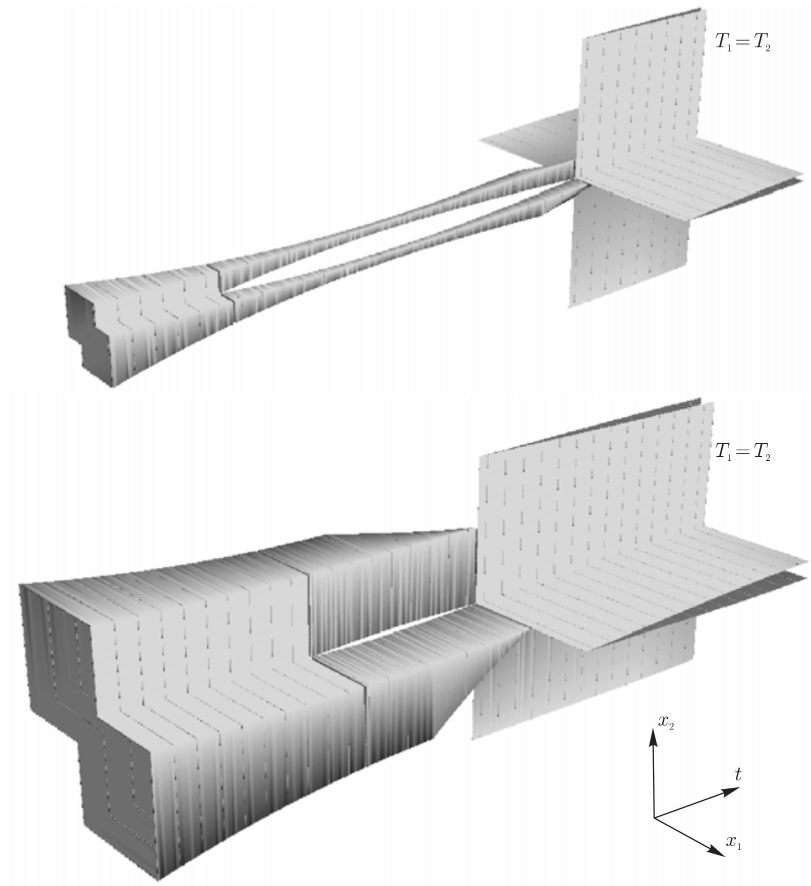

Figure6: Varying advantage of the pursuers, variant 1, equal termination instants: three-dimensional views of the maximal stable bridges $W_{0.526}$ and $W_{3.684}$

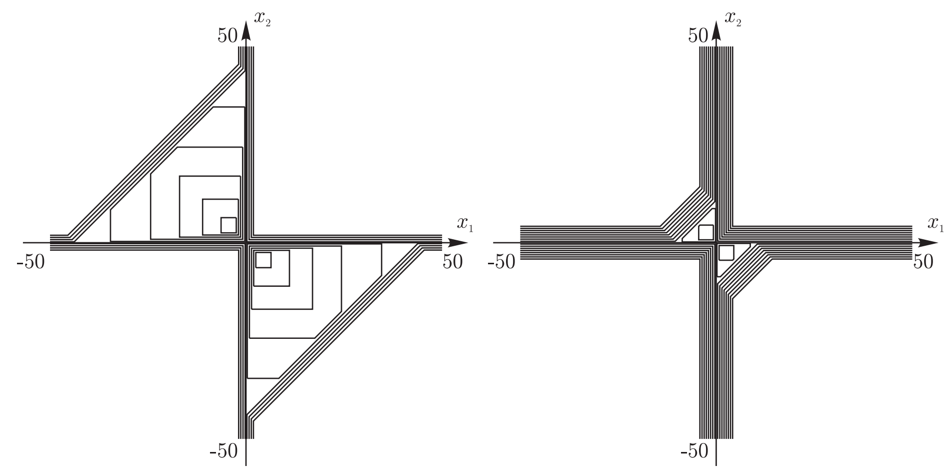

Figure7: Varying advantage of the pursuers, variant 1, equal termination instants: time sections of MSBs $\{W_c(t)\}$ at $t = 12.5$ (*at the left*) and at $t = 16.0$ (*at the right*)

5.2. Variant 2

Let now MSBs in the individual game $P1$–E expand at the beginning of the backward time and further contract ($\eta_1 < 1$, $\eta_1\varepsilon_1 > 1$). Vice versa, in the individual game $P2$–E, let MSBs contract at first and expand further ($\eta_2 > 1$, $\eta_2\varepsilon_2 < 1$). Parameters of the game are taken as follows:

$$\mu_1 = 0.8, \quad \mu_2 = 1.3, \quad \nu = 1, \quad l_{P_1} = 1/20,\ l_{P_2} = 1/0.5, \quad l_E = 1.$$

Termination instants: $T_1 = 15$, $T_2 = 13.5$.

In Fig. 8, t-sections of MSB $W_{0.263}$ are shown for eight instants: $t = 13.5$, 11.95, 9.4, 7.5, 6.45, 5.4, 4.7, 4.45. At the instant $t = T_1 = 15$, the terminal set is taken as a vertical strip with the half-width equal to 0.263.

At the beginning of the backward time, the t-section of MSB is a vertical strip and has growing width. At the instant $t = T_2 = 13.5$, a horizontal strip of half-width 0.263 is added to the vertical one, which is at that instant. With further growing of the backward time, additional curvilinear triangles appear in the II and IV quadrants. Outside them, the horizontal component of the set $W_{0.263}(t)$ con-

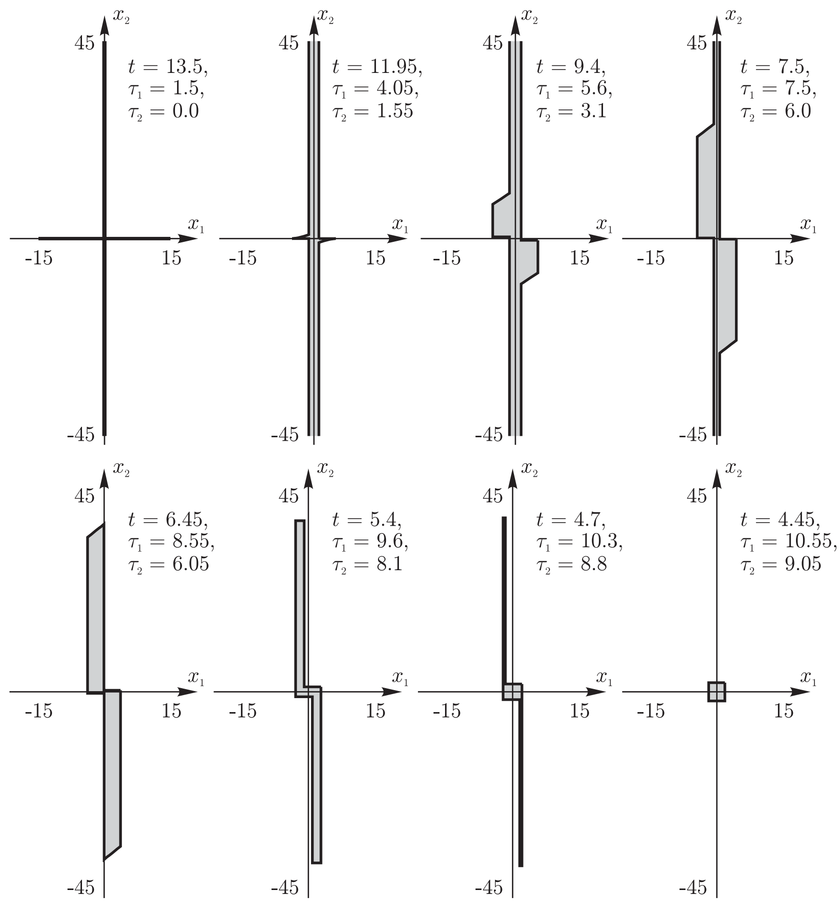

Figure8: Varying advantage of the pursuers, variant 2, different termination instants: t-sections of MSB $W_{0.263}$

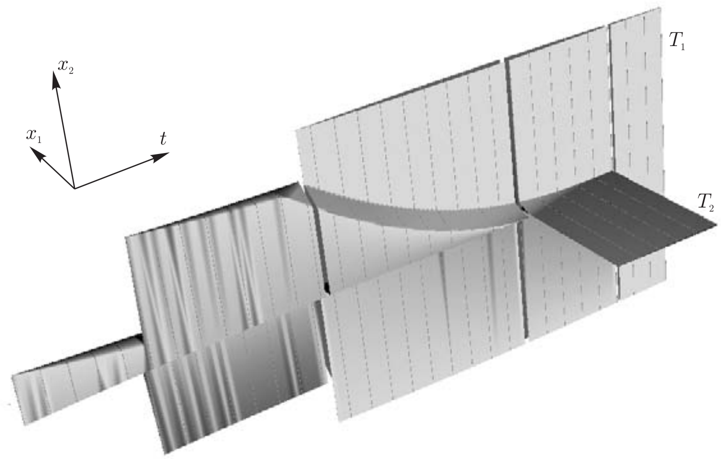

Figure9: Varying advantage of the pursuers, variant 2, different termination instants: a view of the maximal stable bridge $W_{0.263}$ in the three-dimensional space t, x_1, x_2

tracts. At the instant $t = 11.95$, the infinite horizontal component vanishes. Then, some growth in the horizontal direction takes place with high vertical expand of the knobs generated by the curvilinear triangles. Near the instant $t = 9.4$, horizontal increasing is changed by contraction. At the instant $t = 6.45$, the infinite vertical component disappears. Further with growing the backward time, horizontal contraction and vertical dilatation have approximately equal speed. When $t \leq 5.4$, each t-section has two vertical protuberances, which collapse at some instant close to $t = 4.45$. After that, t-sections are rectangles which dilate in the vertical direction and constrict in the horizontal one. At the instant $t = 0.15$, MSB degenerates.

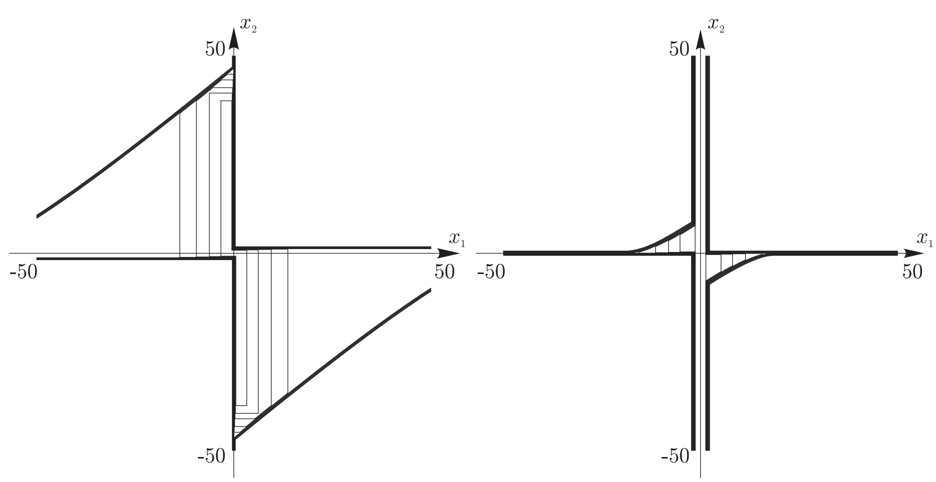

Figure10: Varying advantage of the pursuers, variant 2, different termination instants: time sections of MSBs at $t = 5.85$ (*at the left*) and at $t = 10.0$ (*at the right*)

A three-dimensional view of the set $W_{0.263}$ can be seen in Fig. 9. Time sections of level sets of the value function for two instants $t = 5.85$ and $t = 10.0$ are given in Fig. 10. There are 10 MSBs for c from 0.1 to 1.0 with the step 0.1.

6. Conclusion

The paper deals with numerical investigation of a differential game with two pursuers and one evader. With the help of the standard change of variables, the problem is reduced to a two-dimensional antagonistic game. The difficulty of solution is connected to non-convexity of the terminal payoff function. For typical variants of the game parameters, an analysis of the level sets (Lebesgue sets) of the value function is done. Three-dimensional views of the level sets are given. Here, we do not consider the problem of generating optimal strategies of the players.

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Optimization of Encashment Routs in ATM Network Model

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Abstract The main purpose of this work is to optimize cash flow in case of the encashment process in the ATM network. The solution of these problems is based on some modified algorithms for the Vehicle Routing Problem with Time Windows. A numerical example is considered.

Keywords: ATM network, route optimization, Vehicle Routing Problem with Time Windows.

1. Introduction

Nowadays ATM network and credit cards are the essential parts of modern lifestyle, and one of the most actual problem in the bank's ATM network is optimization of cash flow and organization of uninterrupted work. Serving the ATMs network is a costly task: it takes employees' time to supervise the network and make decisions about cash management and it involves high operating costs (financial, transport, etc.). Banks could reduce their costs applying competent encashment strategy and optimizing encashment routes in ATM network.

For the purpose of reducing bank's costs we could use algorithms for solving Vehicle Routing Problems (VRP). According to (Toth, 2001), the Vehicle Routing Problem is a problem of designing optimal routes for servicing a set of customers by a set of vehicles. The solution of the VRP calls for determination of a set of routes, each route is performed by a single vehicle that starts and ends in its own depot. This set of routes must satisfy the following conditions: all the requirements of the customers are fulfilled, all the operational constraints are satisfied, and the global transportation cost is minimized.

In previous paper (Gubar et al., 2011) we explore one of the modifications of VRPs, the Capacitated Vehicle Routing Problem, where the capacity restrictions for each vehicle are essential. Now we take under consideration the Vehicle Routing Problem with Time Windows (VRPTW) and focus on the fact that additionally each customer is associated with a time interval, called a time window. The service of each customer must start within a given time window. Such additional constraints allow to satisfy the requirements of real-life situations more carefully.

Thus, in this work we consider a problem in which a set of geographically dispersed ATMs with known requirements must be served with a fleet of money collector teams stationed in the depot in such a way as to minimize some distribution objective. We assume that the money collector teams are identical with the equal capacity and must start and finish their routes at the depot.

2. Formulation of the Vehicle Routing Problem with Time Windows

Consider the presentation of the VRPTW, where $V = (0, 1, \ldots, n)$ is the complete set of vertices, each vertex corresponds to an ATM, vertex 0 corresponds to the depot. For each pair of ATMs, or ATMs and the depot, there is an associated cost c_{ij}. Each stop i requires a supply of q_i units from depot 0. A set of M identical vehicles of capacity Q is located at the depot and is used to service the stops; these M vehicles comprise the homogeneous vehicle fleet. It is required that every vehicle route starts and ends at the depot and that the load carried by each vehicle is no greater than Q.

A travel time between ATMs i and j is denoted as t_{ij}. Each stop is associated with a service time σ_i required by a vehicle to visit the ATM and to unload the quantity q_i (we assume $\sigma_0 = 0$). The start time of the service at stop i must be within a given time window $[a_i, b_i]$. A vehicle is permitted to arrive at stop i before the beginning of the time window and wait at no cost until time a_i. Also vehicles are time-constrained at the depot in that each vehicle must leave the depot and return back within the time window $[a_0, b_0]$.

The variable x_{ijk} is $0-1$ binary, it equals to 1 if and only if vehicle k visits stop j immediately after visiting stop i and 0 if not. The continuous variable s_{ik} denotes the time vehicle k begins service at stop i. It is assumed that s_{0k} is the departure time of vehicle k from the depot.

Here we present the formalization of the basic VRPTW problem (Hall, 2003):

$$min \sum_{k=1}^{M} \sum_{i \in V} \sum_{j \in V} c_{ij} x_{ijk}, \tag{1}$$

$$\sum_{k=1}^{M} \sum_{j \in V} x_{ijk} = 1, \qquad i \in V_c, \tag{2}$$

$$\sum_{i \in V_c} q_i \sum_{j \in V} x_{ijk} \leq Q, \qquad k = 1, \ldots, M, \tag{3}$$

$$\sum_{j \in V_c} x_{0jk} \leq 1, \qquad k = 1, \ldots, M, \tag{4}$$

$$\sum_{i \in V} x_{ijk} - \sum_{i \in V} x_{jik} = 0, \qquad j \in V_c,\ k = 1, \ldots, M, \tag{5}$$

$$s_{ik} + \sigma_i + t_{ij} - L(1 - x_{ijk}) \leq s_{jk}, \qquad i \in V,\ j \in V_c,\ k = 1, \ldots, M, \tag{6}$$

$$s_{ik} + \sigma_i + t_{i0} - L(1 - x_{i0k}) \leq b_0, \qquad i \in V_c,\ k = 1, \ldots, M, \tag{7}$$

$$a_i \leq s_{ik} \leq b_i, \qquad i \in V,\ k = 1, \ldots, M, \tag{8}$$

$$x_{ijk} \in \{0, 1\}, \qquad i, j \in V,\ k = 1, \ldots, M. \tag{9}$$

Constraints (2) state that each ATM must be visited exactly once. Constraints (3) are the capacity limitation on the vehicles. Constraints (4) force each vehicle to be used at most once and constraints (5) state that if a vehicle visits ATM, it must also depart from it. Constraints (6) impose that vehicle k cannot arrive at stop j before $s_{ik} + \sigma_i + t_{ij}$, if it travels from i to j. Constraints (7) force each vehicle k to return to the depot before time b_0. The scalar L can be any large number.

Constraints (8) ensure that all time windows are respected and constraints (9) are the integrality constraints.

2.1. Methods for solving VRPTW

General approaches for solving Vehicle Routing Problem with Time Windows could be divided into three groups: exact methods, heuristic and metaheuristic methods.

In exact methods the mixed-integer programming formulation of the VRPTW is solved. Such methods include branch-and-bound, branch-and-cut algorithms, and other techniques for solving integer programming problems. But the VRPTW is considered NP-hard and for problems of practical size computing exact solutions could be too complicated.

Because of the high complexity level of the VRPTW approximate heuristic and metaheuristic methods are of prime importance. Heuristic methods search for not optimal, but approximately optimal high-quality solution in acceptable time.

Heuristics methods for solving VRPTW could be divided into following groups:

1. **Route construction heuristics:** select stops sequentially until a feasible solution has been created. Stops are chosen based on some cost minimization criterion, often subject to the restriction that the selection does not create a violation of vehicle capacity or time window constraints. Among these methods are known:
   - extension to the savings heuristic of Clarke and Wright (Clarke et al., 1964);
   - time-oriented nearest neighbor;
   - Solomons time-oriented sweep heuristic (Solomon, 1987).
2. **Solution Improvement Methods:** based on the concept of iteratively improving the solution to a problem by exploring neighboring vertices.

Metaheuristic methods are the next step in development of heuristic methods. They try overcome the local minima in the searching process, while solution improvement methods stop after finding local solutions in the selected neighborhood. Among metaheuristic methods are known:

- ant colony optimization;
- simulated annealing;
- tabu search;
- genetic algorithms.

In current work we focus on the simulated annealing metaheuristcs for Vehicle Routing Problem with Time Windows and apply it for designing optimal routes for money collector teams.

2.2. Simulated Annealing

Simulated Annealing is an algorithmic approach to solving combinatorial optimization problems (Woch et al., 2009). The name of the algorithm derives from an analogy between solving optimization problems and simulating the annealing of solids. This method accepts search movements that temporarily produces degradations in a current solution found to a problem as a way to escape from local minima.

The simulated annealing algorithm is as follows (Chiang et al., 1996):

Step 1. Obtain an initial feasible solution S for the VRPTW
Step 2. Set the cooling parameters including the initial temperature T, the cooling ratio r, and the epoch length Len
Step 3.
3.1 For $1 < i < Len$ do
3.1.1 Pick a random neighbor solution S'
3.1.2 Let $A = Cost(S') - Cost(S)$
3.1.3 If $A < 0$, then set $S = S'$
3.1.4 If $A > 0$, then set $S = S'$ with probability
3.2. Set $T = rT$
Step 4. Return S

The simulated annealing algorithm starts with the initial feasible solution. To find this initial routes we use a time-oriented nearest-neighbor heuristic method, that belongs to the class of route construction algorithms.

2.3. A Time-Oriented Nearest-Neighbor Heuristic

In terms of our problem of designing optimal routes in ATM network the nearest-neighbor heuristic could be described in the following way. This heuristic starts every route by searching the unrouted ATM "closest" to the bank or the last ATM added without violating feasibility. This search is performed among all the ATMs who can feasibly be added to the end of the emerging route. A new route is started any time the search fails, unless there are no more ATMs to add (Solomon, 1987).

The metric used in this approach tries to account for both geographical and temporal closeness of ATMs. Let the last ATM on the current partial route be ATM i and let j denote any unrouted ATM that could be visited next. Let the metric c_{ij} measures the distance between two ATMs, T_{ij} — the time difference between the end of service at i and the beginning of service at j, and v_{ij} — the urgency of delivery to ATM j:

$$T_{ij} = g_j - (g_i + \sigma_i), \qquad v_{ij} = b_j - (g_i + \sigma_i + t_{ij}), \tag{10}$$

where g_i — the time of beginning servicing ATM i and g_j — the time of beginning servicing ATM j.

$$g_j = max\{a_j, g_i + \sigma_i + t_{ij}\}, \tag{11}$$

where a_i — the lower bound of time window, σ_i — service time of ATM i, and t_{ij} — travel time between ATMs i and j. Then the metric for searching "closest" ATM is:

$$d_{ij} = \delta_1 c_{ij} + \delta_2 T_{ij} + \delta_3 v_{ij}, \qquad \begin{array}{l} \delta_1 + \delta_2 + \delta_3 = 1, \\ \delta_1 \geq 0, \quad \delta_2 \geq 0, \quad \delta_3 \geq 0. \end{array} \tag{12}$$

3. Numerical simulation

Here we represent the application of simulated annealing heuristic for certain ATM network. We assume that the bank has three collector teams with equal vehicle capacity $Q = 12$ cartridges and each ATM requires $q_i = 3$ cartridges. Suppose that money collector teams should service 9 ATMs located at the different subway stations of St.Petersburg: 2 – Tekhnologicheskiy Institut, 3 – Moskovskie Vorota, 4 – Lomonosovskaya, 5 – Vasileostrovskaya, 6 – Prospekt Bol'shevikov, 7 – Ploschad' Lenina, 8 – Narvskaja, 9 – Chkalovskaja and 10 – Sennaja Ploschad'. Distances between ATMs and the Bank are given in the Table 1.

Table1: Distances between ATMs and bank, m

	Bank	2	3	4	5	6	7	8	9	10
Bank	0	3250	6530	9000	5005	10007	6680	7810	7650	3940
2	3250	0	2930	10000	4870	13500	5480	3860	6770	1280
3	6530	2930	0	10120	7940	13070	10610	5410	9180	4050
4	9000	10000	10120	0	13690	6000	11900	14500	15100	10540
5	5005	4870	7940	13690	0	15300	5990	5970	2750	4030
6	10007	13500	13070	6000	15300	0	11100	14560	14600	10480
7	6680	5480	10610	11900	5990	11100	0	9070	4690	6500
8	7810	3860	5410	14500	5970	14560	9070	0	8300	4670
9	7650	6770	9180	15100	2750	14600	4690	8300	0	5010
10	3940	1280	4050	10540	4030	10480	6500	4670	5010	0

Time windows for each ATM are given in the Table 2.

Table2: Time windows, h

i	2	3	4	5	6	7	8	9	10
a_i	10	11	10	11	13	13	10	10	10
b_i	13	18	13	18	18	16	13	18	13

Suppose that working day of money collector teams starts at 10:00 and ends at 18:00, which means that $[a_0, b_0] = [10, 18]$, and average speed of teams is $v_a = 20$ km/h. We also take into account traffic, route features, etc.

We construct the initial solution using the nearest neighbor heuristic with parameters $\delta_1 = 0.4$, $\delta_2 = 0.4$, $\delta_3 = 0.2$. Routes, which were constructed are represented in the Table 3 and Figure 1. The distance travelled on these routes corresponds to 71657 meters. The initial solution was simulated in Maple system.

Table3: The initial solution

Route 1	0-2-10-5-9-0
Route 2	0-3-8-7-0
Route 3	0-4-6-0

Then we apply the simulated annealing heuristic for this initial solution with given parameters of the initial temperature $T = 1000$, the cooling ratio $\alpha = 0.99$,

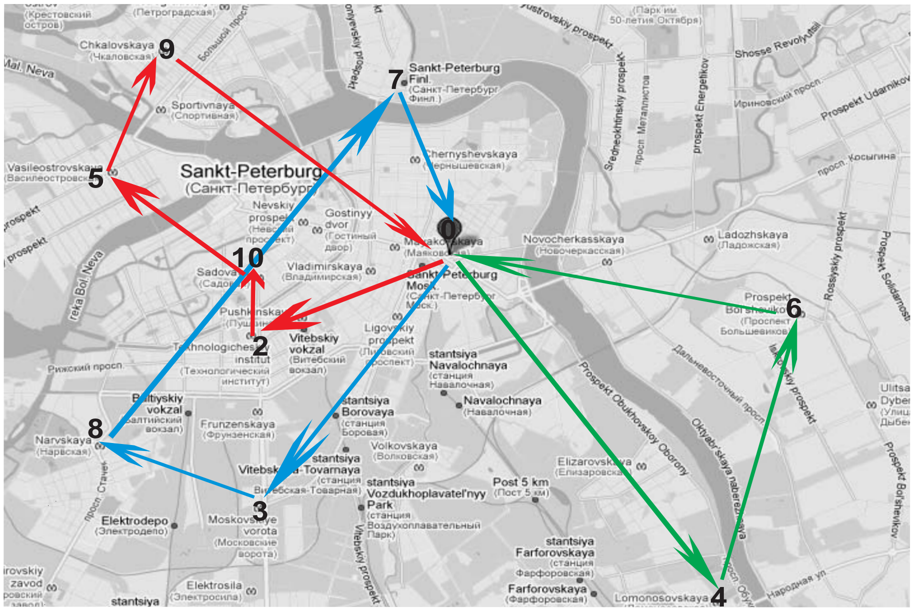

Figure1: The initial solution

and the epoch length $Len = 500$. Routes that we received in Maple system are represented in the Table 4 and Figure 2.

Table4: Solution obtained by simulated annealing algorithm

Route 1	0-10-5-9-7-0
Route 2	0-3-8-2-0
Route 3	0-4-6-0

The optimal solution in the current model consists of three routes, one for each collector team. The first team drives through ATMs 10-5-9-7 (subway stations: Sennaja Ploschad', Vasileostrovskaya, Chkalovskaja, Ploschad' Lenina), the second team goes through ATMs 3-8-2(subway stations: Moskovskie Vorota, Narvskaja, Tekhnologicheskiy Institut) and the third team goes through ATMs 4-6 (subway stations: Lomonosovskaya, Prospekt Bol'shevikov). Every route begins and ends at the bank, vehicle capacity on each route is not exceeded, and time windows are satisfied (see Tables 2 and 5).

The first money collector team returns to the bank at 13:20, the second — at 12:38, the third — at 13:30. That means that the time window of working day is also satisfied.

All ATMs are assigned to a route and total travel costs are minimized. Thus, we got optimal routes for the current request. The distance travelled on this optimal route corresponds to 66147 meters, this is a minimal length of all possible routes for the money collector teams.

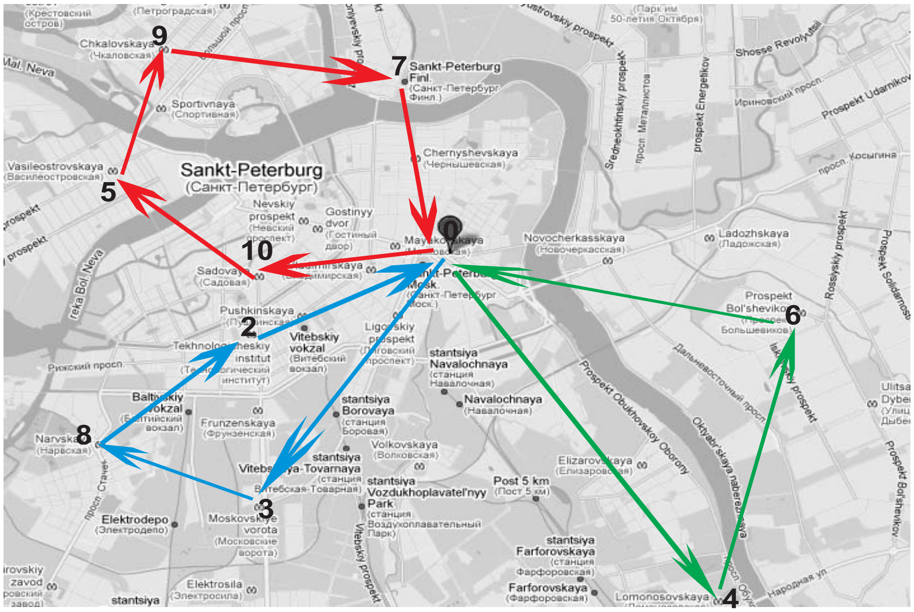

Figure2: Solution obtained by simulated annealing algorithm

Table5: Time of beginning servicing ATMs, h

	Route 1				Route 2			Route 3	
i	10	5	9	7	3	8	2	4	6
g_i	10:12	11:00	11:39	13:00	11:00	11:47	12:28	10:27	13:00

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Stackelberg Strategies for Dynamic Games with Energy Players Having Different Time Durations

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Abstract We consider a system that consists of a major electrical power producer player (Public Power Corporation –PPC) playing in infinite time horizon, and minor players (power producers and consumers) remaining in the system for finite time durations, which time durations are overlapping. We study how they interact among themselves (horizontal interaction), and with the major player respectively (vertical interaction), via their decisions/strategies. We study a deterministic LQ version of the problem in discrete time. In our previous work we employed the Nash equilibrium and we studied the behavior of the system. In this paper we use the Stackelberg equilibrium with the long-term players in the role of the Leader.

Keywords: energy optimization cost, game theory, Stackelberg equilibrium.

1. Introduction

The work presented is motivated by the game between the Public Power Corporation (PPC) referred to as the major player and the many small producers/consumers referred to as the minor ones. We choose to address here the role of the time duration of the minor players (low power producers and consumers) which is small relative to the time horizon of the major player (PPC).

We study a deterministic version of the problem in discrete time. The Nash equilibrium was studied in (Kakogiannis and Papavassilopoulos, 2010) and (Kakogiannis et al., 2010). Here the Stackelberg equilibrium is employed.

We consider the LQ case and since we are interested in strategies that survive in a stochastic framework ((Basar and Olsder, 1999), (Papavassilopoulos, 1982)) we use feedback and closed loop strategies. We provide the solution for the general case using the Ricatti equations. We provide some simple numerical examples for the scalar case. We also assume that all minor players have the same cost function, act during different time periods but for the same duration T. This results to having to solve a system with $T + 1$ equations. Changing the values of the parameters involved we can easily solve each time the system of the Riccati equations and find the optimal controls-decisions and costs in every case for each player.

The Stackelberg solution we employ has a Closed Loop character for the Leader and a Feedback Stackelberg character for the Followers, since Dynamic Programming is used for deriving Followers' decisions. See (Basar and Olsder, 1999), (Simaan and Cruz, 1973a), (Simaan and Cruz, 1973b), (Fudenberg and Tirole, 1991) for more explanations of these concepts. We intend to study this case where Dynamic Programming is used for all the players and thus the Leader uses also a Feedback Stackelberg strategy,in future work.

2. Mathematical Formulation

The state equation is:

$$\begin{gathered} x_{k+1} = Ax_k + B_0 u_k + B_1 u_{1k} + B_2 u_{2k} + B_3 u_{3k} + \\ + B_4 u_{4k} + B_5 u_{5k} \\ k = 0{,}1, 2, 3 \ldots \end{gathered} \tag{1}$$

where x_k is the state, u_k is the control of the long term player (PPC) - *Leader*, u_{ik} is the control of the minor players (clients or producers) – *Followers* at the i-th year remaining in the system ($i = 1 - 5$). For example, a minor player who enters the game at time k will use controls u_{1k}, u_{2k+1}, u_{3k+2}, u_{4k+3}, u_{5k+4}, corresponding to times k, $k+1$, $k+2$, $k+3$, $k+4$. A, B_i are given matrices of appropriate dimensions. If the players are six the state equation is

$$\begin{gathered} x_{k+1} = Ax_k + B_0 u_k + B_1 u_{1k} + B_2 u_{2k} + B_3 u_{3k} + \\ + B_4 u_{4k} + B_5 u_{5k} + B_6 u_{6k} \\ k = 0{,}1, 2, 3 \ldots \end{gathered}$$

and so on for 7 and 8 players.

The quadratic costs of the major player J_0 and the minor players ($J_1 l$) who act in the interval l and $(l+4)$ are:

$$\begin{gathered} J_0 = \sum_0^{\infty} (x_k^T Q_0 x_k + u_k^T R_0 u_k) \\ J_1 = \sum_{k=0}^{4} (x_{k+l+1}^T Q_f x_{k+l+1} + u_{(k+1)(l+k)}^T R_f u_{(k+1)(l+k)}) + x_l^T Q_f x_l \end{gathered} \tag{2}$$

The Q's are symmetric non negative matrices and the R's are symmetric positive defined matrices which are known. In our case we consider A, B, Q, and R constant. We will consider linear strategies for all the players, which will be derived as follows. Let $u_k = L_0 x_k$ be the Leader's strategy who is assumed to play this linear stationary strategy throughout the game. Each minor player (Follower) who acts for a period of length five, faces a Linear Quadratic problem where besides the Leader, several other minor players (Followers) are also present. We assume that they all play linear strategies in which case we can write the Ricatti equations that solve the minor player's problem. To derive the equations that provide the Li's of the minor player we proceed as follows. Consider for example the minor player who enters the calendar year 30 ($k = 30$). He sees the following system (7)-(13) where in this first equation (7) he acts as first year consumer/producer. The consumers/producers who entered earlier act with the fixed laws $L_2 x_{30}$, $L_3 x_{30}$, $L_4 x_{30}$, $L_5 x_{30}$

$$\begin{aligned} x_{k+1} &= (A + B_0 L_0 + BL_2 + BL_3 + BL_4 + BL_5) x_k + Bu_{1,k} \\ &= A_1 x_k + Bu_{1,k} \end{aligned} \tag{3}$$

Similarly when he is at the second year he sees the following system

$$\begin{aligned} x_{k+2} &= (A + B_0L_0 + BL_1 + BL_3 + BL_4 + BL_5)x_k + Bu_{2,k+1} \\ &= A_2x_{k+1}{+}Bu_{2,k+1} \end{aligned} \tag{4}$$

and the producers/consumers who entered earlier act with the fixed laws L_2x_{30}, L_3x_{30}, L_4x_{30}, L_5x_{30} and so on.

Thus the whole system of equations that the minor player (Follower) who entered the calendar year $k = 30$ and stays for five years sees, is:

$$\begin{aligned} &x_{k+1} = (A + B_0L_0 + BL_2 + BL_3 + BL_4 + BL_5)x_k + Bu_{1,k} \\ &= A_1x_k + Bu_{1,k} \end{aligned} \tag{5}$$

$$\begin{aligned} &x_{k+2} = (A + B_0L_0 + BL_1 + BL_3 + BL_4 + BL_5)x_{k+1} + Bu_{2,k+1} \\ &= A_2x_{k+1} + Bu_{2,k+1} \end{aligned} \tag{6}$$

$$\begin{aligned} &x_{k+3} = (A + B_0L_0 + BL_1 + BL_2 + BL_4 + BL_5)x_{k+2} + Bu_{3,k+2} \\ &= A_3x_{k+2} + Bu_{3,k+2} \end{aligned} \tag{7}$$

$$\begin{aligned} &x_{k+4} = (A + B_0L_0 + BL_1 + BL_2 + BL_3 + BL_5)x_{k+3} + Bu_{4,k+3} \\ &= A_4x_{k+3} + Bu_{4,k+3} \end{aligned} \tag{8}$$

$$\begin{aligned} &x_{k+5} = (A + B_0L_0 + BL_1 + BL_2 + BL_3 + BL_4)x_{k+4} + Bu_{5,k+4} \\ &= A_5x_{k+4} + Bu_{5,k+4} \end{aligned} \tag{9}$$

For this system of equations (5)-(9) and the cost

$$J_{30} = \sum_{k=0}^{4}(x^T_{k+30+1}Q_f x_{k+30+1} + u^T_{(k+1)(30+k)}R_f u_{(k+1)(30+k)}) + x^T_{30}Q_f x_{30} \tag{10}$$

we derive the optimal policy by employing the Ricatti equations. It holds:

$$u_{1,k} = L_1x_k, \quad u_{2,k+1} = L_2x_{k+1}, \quad u_{3,k+2} = L_3x_{k+2}$$
$$u_{4,k+3} = L_4x_{k+3}, \quad u_{5,k+4} = L_5x_{k+4}$$

where the Li's are given by the following system of equations.

$$L_1 = -(B^TK_2B + R)^{-1}B^TK_2A_1 \tag{11}$$
$$K_1 = A_1^T(K_2 - K_2B(B^TK_2B + R)^{-1}B^TK_2)A_1 + Q_f \tag{12}$$
$$L_2 = -(B^TK_3B + R)^{-1}B^TK_3A_2 \tag{13}$$
$$K_2 = A_2^T(K_3 - K_3B(B^TK_3B + R)^{-1}B^TK_3)A_2 + Q_f \tag{14}$$
$$L_3 = -(B^TK_4B + R)^{-1}B^TK_4A_3 \tag{15}$$
$$K_3 = A_3^T(K_4 - K_4B(B^TK_4B + R)^{-1}B^TK_4)A_3 + Q_f \tag{16}$$
$$L_4 = -(B^TK_5B + R)^{-1}B^TK_5A_4 \tag{17}$$
$$K_4 = A_4^T(K_5 - K_5B(B^TK_5B + R)^{-1}B^TK_5)A_4 + Q_f \tag{18}$$
$$L_5 = -(B^TK_6B + R)^{-1}B^TK_6A_5 \tag{19}$$
$$K_5 = A_5^T(K_6 - K_6B(B^TK_6B + R)^{-1}B^TK_6)A_5 + Q_f \tag{20}$$
$$K_6 = Q_f \tag{21}$$

Since the other Followers use a similar rational, the Li's used by them and are present in the of (5)-(9) are identified with the Li's of the player under consideration derived in (11)-(21). The total cost of a minor player who entered the system at year 30 is:

$$J_3 0^* = x_{30}^T K_1 x_{30} \tag{22}$$

Notice that we consider linear no memory strategies. We know that may exist other solutions, which are not necessarily linear and may have memory. We know nonetheless (Selten and (Basar and Olsder, 1999)) that these solutions disappear in the presence of noise.

The Leader's cost ,can be found as follows:

$$\overline{A} = A + B_0 L_0 + BL_2 + BL_3 + BL_4 + BL_5 \tag{23}$$

$$x_k = (\overline{A})^k x_0 \tag{24}$$

$$J_0 = \sum x_0^T (\overline{A}^T)^k Q_0 (\overline{A})^k x_0 + (L_0 x_k)^T R_0 (L_0 x_k) \tag{25}$$

$$J_0 = x_0^T \sum_{k=0}^{\infty} (\overline{A}^T)^k (Q_0 + L_0^T R_0 L_0)(\overline{A}^T)^k x_0 \tag{26}$$

The Leader's problem is to minimize (26) subject to the constraints (11)-(21) and (24). It is obviously a difficult nonlinear programming problem. It should be noted that the Leader's strategy is not derived by the Dynamic Programming Algorithm and thus it cannot be considered as a Feedback Stackelberg Strategy, as defined in (Simaan and Cruz, 1973a), (Simaan and Cruz, 1973b), (Basar and Olsder, 1999). On the other hand, the Followers' Strategies obey Dynamic Programming since they were derived using the Ricatti equations formalism , and thus can be called Feedback Stackelberg Strategies.

3. Numerical Study

In this section we present some numerical results for the scalar case and study the optimal cost of the Leader for several values of the parameters. We consider the matrices A, B, Q_0, Q_f, R as constant scalars a, b, q_0, q_f, r. We take the R's and the B's to be equal to 1. We also take the initial condition equal to 1.

After some transformations we created the following scalar equations ,where the xi's, stand for the Ki's, i=0,1,...5:

$$x_5 = q_f \tag{27}$$

$$x_4 = q_f + \overline{A}^2(x_5 + x_5^2) \tag{28}$$

$$x_3 = q_f + \overline{A}^2(x_4 + x_4^2) \tag{29}$$

$$x_2 = q_f + \overline{A}^2(x_3 + x_3^2) \tag{30}$$

$$x_1 = q_f + \overline{A}^2(x_2 + x_2^2) \tag{31}$$

$$\overline{A} = \frac{a}{1 + S + x_0} \tag{32}$$

$$S = x_1 + x_2 + x_3 + x_4 + x_5 \tag{33}$$

$$x_0 = \frac{a}{\overline{A}} - 1 - S \tag{34}$$

$$J_0 = \sum(q_0 x_k^2 + u_k^2) \tag{35}$$

$$J_0 = (q_0 + x_0^2)\frac{1}{1 - \overline{A}^2} \tag{36}$$

The quantity $\overline{A}$ in (32) is actually the closed loop matrix of the system which has to be stable, i.e. $-1 < \overline{A} < 1$ .The problem for the Leader is to minimize J_0 subject to the constrains (27)-(32) where all the x_0, x_1, x_2, x_3, x_4, x_5, $\overline{A}$ are unknowns.

A way of solving this system is to use Lagrange Multipliers and append the constrains to the cost.

We will present a quicker way based on a plot of the cost of the Leader versus the policy gain, (here it is the x_0), from where the optimal policy gain and cost of the Leader can be found. For fixed values of a, q_0, q_f and initial condition 1 we do the following. We take a value of $\overline{A} \in (-1, 1)$ for

example $\overline{A} = 0.3$. For (27)-(32) we find the values for x_0 and J_0. We do that for several values of $\overline{A} \in (-1, 1)$ and plot $J_0 - x_0$.

We present some runs and numerical results. Then we present 4 plots of J_0 (vertical) versus x_0 (horizontal) from which the best choice of the Leader's gain and his best cost are easily found. The values of a, q_0 and q_f used in these plots are given below.

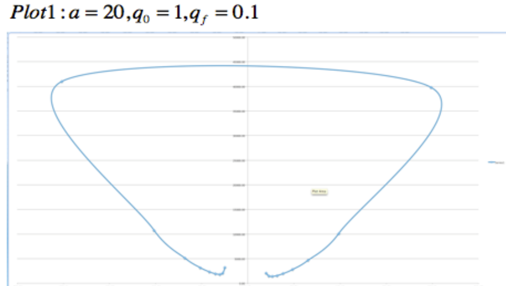
Numerical Results - Plots:

Case of plot 1:

α	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
qo	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
qf	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
A	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10
x1	0.54	0.33	0.22	0.17	0.14	0.12	0.11	0.10	0.10
x2	0.39	0.28	0.21	0.17	0.14	0.12	0.11	0.10	0.10
x3	0.28	0.23	0.19	0.16	0.14	0.12	0.11	0.10	0.10
x4	0.19	0.17	0.15	0.14	0.13	0.12	0.11	0.10	0.10
x5	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
xo	-24.73	-27.11	-30.44	-35.07	-41.64	-51.58	-68.21	-101.52	-201.50
S	1.51	1.11	0.87	0.73	0.64	0.58	0.54	0.52	0.50
Jo	3224.04	2043.58	1819.38	1922.76	2313.35	3168.72	5113.81	10736.41	41015.19

α	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00	20.00
qo	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
qf	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
A	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
x1	0.10	0.10	0.11	0.12	0.14	0.17	0.22	0.33	0.54
x2	0.10	0.10	0.11	0.12	0.14	0.17	0.21	0.28	0.39
x3	0.10	0.10	0.11	0.12	0.14	0.16	0.19	0.23	0.28
x4	0.10	0.10	0.11	0.12	0.13	0.14	0.15	0.17	0.19
x5	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
xo	198.50	98.48	65.12	48.42	38.36	31.60	26.70	22.89	19.71
S	0.50	0.52	0.54	0.58	0.64	0.73	0.87	1.11	1.51
Jo	39799.48	10103.81	4661.63	2792.01	1963.17	1561.96	1399.58	1458.82	2050.86

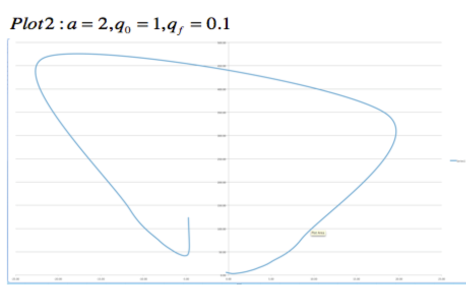
Plot1 : $a = 20, q_0 = 1, q_f = 0.1$



Case of plot 2:

α	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
qo	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
qf	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
A	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10
x1	0.54	0.33	0.22	0.17	0.14	0.12	0.11	0.10	0.10
x2	0.39	0.28	0.21	0.17	0.14	0.12	0.11	0.10	0.10
x3	0.28	0.23	0.19	0.16	0.14	0.12	0.11	0.10	0.10
x4	0.19	0.17	0.15	0.14	0.13	0.12	0.11	0.10	0.10
x5	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
xo	-4.73	-4.61	-4.73	-5.07	-5.64	-6.58	-8.21	-11.52	-21.50
S	1.51	1.11	0.87	0.73	0.64	0.58	0.54	0.52	0.50
Jo	123.01	61.69	45.84	41.65	43.77	52.77	75.16	139.24	468.12

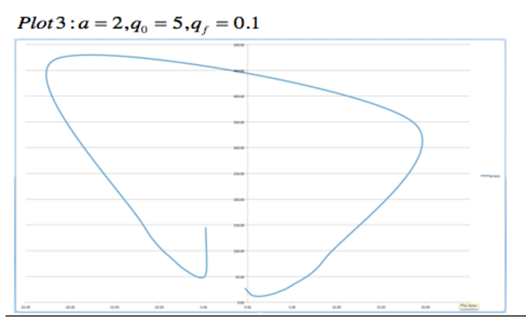
α	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
qo	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
qf	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
A	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
x1	0.10	0.10	0.11	0.12	0.14	0.17	0.22	0.33	0.54
x2	0.10	0.10	0.11	0.12	0.14	0.17	0.21	0.28	0.39
x3	0.10	0.10	0.11	0.12	0.14	0.16	0.19	0.23	0.28
x4	0.10	0.10	0.11	0.12	0.13	0.14	0.15	0.17	0.19
x5	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
xo	18.50	8.48	5.12	3.42	2.36	1.60	0.98	0.39	-0.29
S	0.50	0.52	0.54	0.58	0.64	0.73	0.87	1.11	1.51
Jo	346.55	75.98	29.95	15.10	8.75	5.57	3.86	3.21	5.69



Case of plot 3:

α	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
qo	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00
qf	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
A	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10
x1	0.54	0.33	0.22	0.17	0.14	0.12	0.11	0.10	0.10
x2	0.39	0.28	0.21	0.17	0.14	0.12	0.11	0.10	0.10
x3	0.28	0.23	0.19	0.16	0.14	0.12	0.11	0.10	0.10
x4	0.19	0.17	0.15	0.14	0.13	0.12	0.11	0.10	0.10
x5	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
xo	-4.73	-4.61	-4.73	-5.07	-5.64	-6.58	-8.21	-11.52	-21.50
S	1.51	1.11	0.87	0.73	0.64	0.58	0.54	0.52	0.50
Jo	144.06	72.80	53.68	47.90	49.10	57.53	79.56	143.41	472.16

α	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
qo	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00
qf	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
A	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
x1	0.10	0.10	0.11	0.12	0.14	0.17	0.22	0.33	0.54
x2	0.10	0.10	0.11	0.12	0.14	0.17	0.21	0.28	0.39
x3	0.10	0.10	0.11	0.12	0.14	0.16	0.19	0.23	0.28
x4	0.10	0.10	0.11	0.12	0.13	0.14	0.15	0.17	0.19
x5	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
xo	18.50	8.48	5.12	3.42	2.36	1.60	0.98	0.39	-0.29
S	0.50	0.52	0.54	0.58	0.64	0.73	0.87	1.11	1.51
Jo	350.59	80.15	34.34	19.86	14.08	11.82	11.70	14.32	26.74

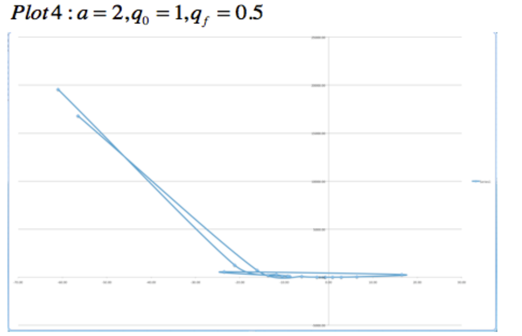


Case of plot 4:

α	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
qo	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
qf	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
A	-0.90	-0.80	-0.70	-0.60	-0.50	-0.40	-0.30	-0.20	-0.10
x1	46.66	10.87	3.33	1.46	0.90	0.68	0.58	0.53	0.51
x2	7.07	3.56	1.95	1.21	0.85	0.68	0.58	0.53	0.51
x3	2.39	1.74	1.29	0.99	0.79	0.66	0.58	0.53	0.51
x4	1.11	0.98	0.87	0.77	0.69	0.62	0.57	0.53	0.51
x5	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
xo	-60.94	-21.15	-11.80	-9.27	-8.73	-9.14	-10.48	-13.63	-23.53
S	57.72	17.65	7.94	4.93	3.73	3.14	2.81	2.63	2.53
Jo	19553.76	1245.37	275.05	135.73	102.87	100.59	121.78	194.49	560.29

α	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00	2.00
qo	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
qf	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
A	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
x1	0.51	0.53	0.58	0.68	0.90	1.46	3.33	10.87	46.66
x2	0.51	0.53	0.58	0.68	0.85	1.21	1.95	3.56	7.07
x3	0.51	0.53	0.58	0.66	0.79	0.99	1.29	1.74	2.39
x4	0.51	0.53	0.57	0.62	0.69	0.77	0.87	0.98	1.11
x5	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50
xo	16.47	6.37	2.85	0.86	-0.73	-2.60	-6.09	-16.15	-56.50
S	2.53	2.63	2.81	3.14	3.73	4.93	7.94	17.65	57.72
Jo	275.00	43.34	10.05	2.08	2.04	12.12	74.62	727.30	16806.52

*Plot*4 : $a = 2, q_0 = 1, q_f = 0.5$



4. Conclusions

In our future search we intend to use the matrix Stackelberg equilibrium model for the players and not only scalar form. Of interest is also the case where the time duration of the short time players is a random variable taking values in a certain interval. Similarly we can consider cases where the appearance of a small duration player at each instant of time is itself a random event.

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The Application of Stochastic Cooperative Games in Studies of Regularities in the Realization of Large-scale Investment Projects

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Abstract The paper considered possible way of implementations of classical cooperative games with transferable utility. This way is based on the assumption, that the utility of coalitions (the value of the characteristic function of the game) are stochastic values. The given class of games is offered to be called stochastic cooperative games. The main attention is placed on possible approaches to the definition of superadditivity for stochastic cooperative games. Also was considered the possible approaches to the definition of concept of imputations and core for stochastic cooperative games. One of the possible areas of practical using of stochastic cooperative games are economic researches of the processes of large investment project, including projects with international participation.

Keywords: stochastic cooperative games, superadditivity, imputation

1. Introduction

In today's economic situation the research of the patterns of occurrence and the subsequent development of large-scale investment projects is becoming more and more essential. Such projects are often characterized by a rather diverse composition of participants in terms of scale, and in terms of organizational and legal forms.

In recent years projects of public-private partnerships, as well as large-scale interstate projects, which involve diverse and disparate investors are becoming more and more important. The traditional classic studies in the area of investment are primarily focused on the problems of their evaluation, as well as on issues of risk management and the uncertainties that exist objectively at all stages of the implementation of large-scale investment projects (activities).

At the same time, quite an interesting subject for research is the study of cooperative effects, inevitably appearing in the formation and subsequent development of coalitions of investors - especially in situations where the parties of these coalitions have differences not only in the organizational or material parameters, but also in economic interests. In such cases mathematical models and methods that give us an opportunity to analyze patterns of major groups (coalitions) of investors can be widely used.

2. Basic definitions

Is seems natural to apply methods of cooperative games as tools for solving these problems. Simplified situation in which we study the possibility of association of investors in terms of implementing a large-scale investment project, we can describe the classical cooperative game with transferable utility (I, v) in the following way

- $v(i)$ – incomes, which individual investors $i \in 1..m$ can gain if they act separately;
- $v(S)$ – incomes of all possible coalitions, which the participants can form ($S \subset 2^I$).

Usage of the term "large-scale" in "large-scale investment project" is explained, first of all, by the wish to highlight the need for joint efforts of all stakeholders of the economic subsystem to implement the project. Therefore, the utility of the largest (major) coalition $v(I)$, which is formed with all the participants $I = \{1..m\}$, equals the utility of project realization.

Among the "principal" of the disadvantages of this purely theoretical, limited and primitive model we can highlight the following: the supposition of representing income of individual participants and their various coalitions in the form of deterministic values. A more plausible, and therefore more attractive is the assumption that these profits are random variables $\tilde{v}(S\}$ with some known distribution functions.

$$F_{\tilde{v}(s)}(x) = P\{\tilde{v}(s) \leq x\}.$$

Thus we realize that we need to modify classical cooperative games in a way that a factor of randomness in values of characteristic features can be considered in them. Thus under stochastic cooperative game (SCG), we understand a pair of sets $\Gamma = (I,\ \tilde{v})$, where

- $I = \{1..m\}$ – is the set of participants;
- $\tilde{v}(S)$ – random variables with determined density functions $p_{\tilde{v}(s)}(x)$, which are interpreted as incomes (utilities), which coalitions $S \subset I$ get.

Among works mentioning problems of stochastic cooperative games can be listed (Amir; Baranova and Petrosjan; Dutta, 1995; Haller and Lagunoff, 2000; Herings and Peeters, 2004). At the same time we'll notice this term is used in different sense in this work.

Under this approach, we should pay more attention to how we are going to integrate such concepts as superadditivity, convexity, imputation, core into stochastic cooperative games.

3. Superadditivity in stochastic cooperative games

Almost all the courses on cooperative games begin with a definition of superadditivity properties of the games. Under superadditivity we understand such games, in which coalitions S and T satisfy the condition condition

$$v(S \cup T) \geq v(S) + v(T).$$

In other words, the utility of a combined coalition is not less than the sum of utilities of its parts. It is quite natural and logical to attempt to introduce a similar term for stochastic cooperative games. Here, taking into account the fact that utilities

$\tilde{v}(S)$ are random variables, we get at least two approaches to the definition of superadditivity.

The first is based on expected values of $\tilde{v}(S)$. According to this approach a game will be superadditive if any coalition $S, T \subset I$ $(S \cap T = \emptyset)$ satisfies the following condition:

$$E\{\tilde{v}(S \cup T)\} \geq E\{\tilde{v}(S)\} + E\{\tilde{v}(T)\}. \tag{1}$$

In this interpretation of superadditivity we substitute random utilities $\tilde{v}(S)$ with their expectations $E\{\tilde{v}(S)\}$, which essentially means a return to traditional deterministic games from the stochastic cooperative ones. Disadvantages of this approach are connected to the fact that the expatiation (weighted average) is generally not the only characteristic of a random variable.

On the other hand the definition of this term may be based not on mathematical expectations, but on the distribution functions. The game will be called superadditive if there is a probability α that any coalition $S, T \subset I$ $(S \cap T = \emptyset)$ satisfies the condition:

$$P\{\tilde{v}(S \cup T) \geq \tilde{v}(S) + \tilde{v}(T)\} \geq \alpha. \tag{2}$$

We would call a game strictly superadditive if the condition (2) is fulfilled for any α.

It is obvious that whether this condition will be fulfilled depends on the type of function of distribution of random variables $\tilde{v}(S)$, $\tilde{v}(T)$, $\tilde{v}(S \cup T)$.

Regarding this we should draw our attention to another important property of stochastic cooperative games. It is known that the classical cooperative game with transferable utility is called inessential if for any coalition $S \subset I$

$$v(S) = \sum_{i \in S} v(i).$$

At the same time, when the values of the characteristic functions in the game (utilities) $\tilde{v}(i)$ are random variables with some distribution functions $F_{\tilde{v}(i)}(x)$, then even the simple addition of them to the emergence of a new random variable $\sum\limits_{i \in S} \tilde{v}(i)$ with its own distribution function, which may be complexly associated with functions $F_{\tilde{v}(i)}(x)$.

Generally we can highlight the following basic situations in stochastic cooperative games that may arise in the proves of creating of their characteristic functions:

- utility of coalitions S and T when they are merged into coalition $S \cup T$ is a new random variable $\tilde{v}(S \cup T)$ with the distribution function $F_{\tilde{v}(S \cup T)}(x)$, which corresponds to the "meaning" of the situation (we have an analogical case when the meanings of $v(S)$, $v(T)$ and $v(S \cup T)$ are considered exogenous);
- the utility of the merged coalition $S \cup T$ is a sum of utilities of coalitions S and T (this situation is only interesting from the point of view of cooperative stochastic games)

In the future in order to distinct the mentioned types of characteristic functions we would denote the utility of a merged coalition in the first case as $\tilde{v}(S \cup T)$, in the second as $\tilde{v}^{+}(S \cup T)$.

It is also important to note that when we add up utilities of coalitions, we get two different situations, namely:

- random variables $\tilde{v}(i)$ (individual utilities of the players) are independent;
- random variables $\tilde{v}(i)$ (individual utilities of the players) are not independent.

Of course, we can not exclude the possibility that there exist both types of coalition formation in the game: the first type is "full" association, which leads to a qualitatively new utility $\tilde{v}(S \cup T)$ or coalition formed by the agreement of summing utilities $\tilde{v}^{+}(S \cup T)$. This raises an interesting challenge of matching these values. In terms of economics it can be interpreted as a problem of how closely should economic agents merge. For example, if there should be a complete takeover of one company by another, or simply a cartel agreement between them.

Let us consider in more detail the concept of superadditivity (in the sense of definition (2)) for stochastic cooperative games. Suppose that some player i of a stochastic game has an individual utility $\tilde{v}(i)$, and the j player has utility $\tilde{v}(j)$. Then in order to verify the superadditivity condition (2) at a certain level of probability α in terms of co-operation of the "summing utilities" we would have to compare of the sum of $(1-\alpha)$-quantiles of random variables $\tilde{v}(i)$ and $\tilde{v}(j)$ with $(1-\alpha)$-quantile of a random variable $\tilde{v}^{+}(i \cup j) = \tilde{v}(i) + \tilde{v}(j)$. We introduce the notation:

$$v_{1-\alpha}(i) = F^{-1}_{\tilde{v}(i)}(1-\alpha), \quad F_{\tilde{v}(i)}(x) = P\{\tilde{v}(i) \le x\}, \tag{3}$$

$$v_{1-\alpha}(j) = F^{-1}_{\tilde{v}(j)}(1-\alpha), \quad F_{\tilde{v}(j)}(x) = P\{\tilde{v}(j) \le x\}, \tag{4}$$

$$v^{+}_{1-\alpha}(i \cup j) = F^{-1}_{\tilde{v}^{+}(i \cup j)}(1-\alpha), \quad F_{\tilde{v}^{+}(i \cup j)}(x) = P\{\tilde{v}(i) + \tilde{v}(j) \le x\}. \tag{5}$$

From the content point of view $v_{1-\alpha}(i)$ represents the level of utility which the player i would not achieve with probability $1-\alpha$ (he will achieve less with probability α). In terms of contemporary risk-management $v_{1-\alpha}(i)$ is VaR (Value At Risk) of a stochastic variable of utility of the player$i$ (Figure 1).

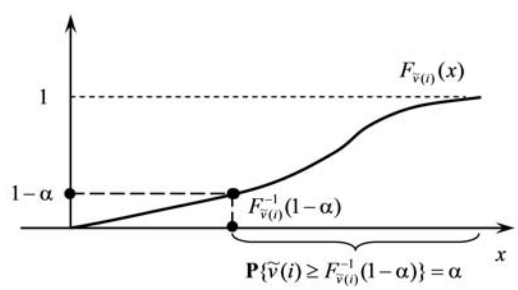


Figure1: VaR (Value At Risk) of the utility of player i in a stochastic cooperative game

To illustrate the potential of research in stochastic superadditivity games we are going to focus on one important special case. Namely, consider a game in which utilities $\tilde{v}(i)$ are random variables distributed according to the normal law with parameters m_i and σ_i^2 ($\tilde{v}(i) \in N(m_i, \sigma_i^2)$. This assumption is consistent with the objective economic characteristics of values simulated, realizations of which we can describe as symmetric intervals $\pm 3\sigma_i$ located with respect to some expected average m_i.

It is obvious that parameters of distribution of a random variable $\tilde{v}^+(i \cup j)$ are determined by the parameters $\tilde{v}(j)$ and $\tilde{v}(j)$. In case of normally distributed individual utilities we have

$$v_{1-\alpha}(i) = m_i + \sigma_i \cdot \Phi^{-1}(1-\alpha), \tag{6}$$

where $\Phi(x) = \frac{1}{2\pi} \cdot \int\limits_{-\infty}^{x} e^{-\frac{t^2}{2}} dt$ is Laplace's integral, therefore distribution function for $\tilde{v}(i)$ can be written down as

$$F_{\tilde{v}(i)}(x) = \Phi\left(\frac{x - m_i}{\sigma_i}\right). \tag{7}$$

Under the assumptions we've made $\tilde{v}^+(i \cup j)$ is also normally distributed

$$\tilde{v}^+(i \cup j) \in N(m_i + m_j, \sqrt{\sigma_i^2 + \sigma_j^2}). \tag{8}$$

Then

$$v^+_{1-\alpha}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j)) =$$

$$= (m_i + m_j + \sqrt{\sigma_i^2 + \sigma_j^2} \cdot \Phi^{-1}(1-\alpha)) - (m_i + \sigma_i \cdot \Phi^{-1}(1-\alpha) + m_j + \sigma_j \cdot \Phi^{-1}(1-\alpha)) =$$

$$= (\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j)) \cdot \Phi^{-1}(1-\alpha). \tag{9}$$

Knowing that $\sigma_i \geq 0$ and $\sigma_j \geq 0$, we have

$$\sqrt{\sigma_i^2 + \sigma_j^2} \leq \sigma_i + \sigma_j \tag{10}$$

or

$$\sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j) \leq 0. \tag{11}$$

Thus, taking into consideration that $\Phi^{-1}(1-\alpha) \leq 0$ when $\alpha \geq 0.5$ and $\Phi^{-1}(1-\alpha) \geq 0$ when $\alpha \leq 0.5$, we get

$$v^+_{1-\alpha}(i \cup j) \geq v_{1-\alpha}(i) + v_{1-\alpha}(j) \quad \text{when} \quad \alpha \geq 0.5, \tag{12}$$

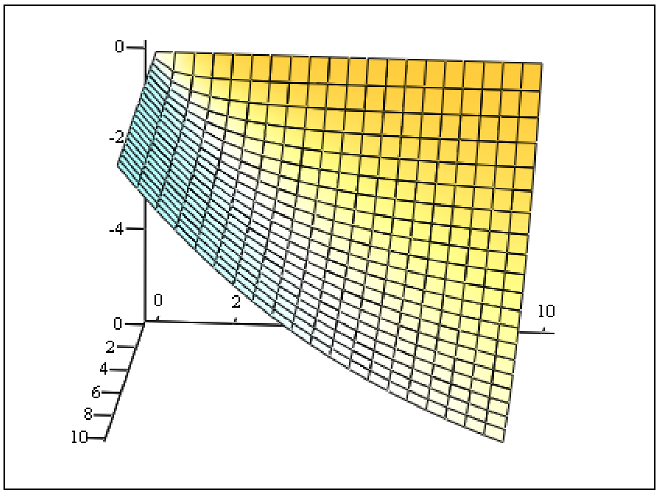
$$v^+_{1-\alpha}(i \cup j) \leq v_{1-\alpha}(i) + v_{1-\alpha}(j) \quad \text{when} \quad \alpha \leq 0.5. \tag{13}$$

It can be deducted from the condition (12) that if utilities of the players i and j are normally distributed, then it is rational for them to behave cooperatively by adding up the utilities (values of the characteristic function). The effect of such an association (excess of the VaR of the sum of utilities over sum of VaR-s with the level of probability $\alpha \geq 0.5$)

$$v^+_{1-\alpha}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j)) = \Phi^{-1}(1-\alpha) \cdot \left\lfloor \sqrt{\sigma_i^2 + \sigma_j^2} - (\sigma_i + \sigma_j) \right\rfloor. \tag{14}$$

Taking into consideration that the value $\Phi^{-1}(1-\alpha)$ is constant for a fixed level of α, we deduct that in the formula (14) the value of "the effect from adding up utilities" is determined by multiplier $\sqrt{\sigma_i^2+\sigma_j^2}-(\sigma_i+\sigma_j)$, which depends on standard deviations σ_i, σ_j: when σ_i and σ_j grow, as $\Phi^{-1}(1-\alpha) \le 0$ when $\alpha \ge 0.5$ $(1-\alpha \le 0.5)$, $v^+_{1-\alpha}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j))$ grows.

Let us consider behavior of multiplier $\sqrt{\sigma_i^2+\sigma_j^2}-(\sigma_i+\sigma_j)$ in a more detailed way. The surface plot which corresponds to it when $\sigma_i, \sigma_j \in [0,\ 10]$ is represented on Figure 2.



vij

Figure2: Surface plot of $\sqrt{\sigma_i^2+\sigma_j^2}-(\sigma_i+\sigma_j)$

Let us denote $\sigma_j = \lambda \cdot \sigma_i$. At the same time without loss of generality we can assume that σ_i and σ_j chosen in such a way that $\sigma_i < \sigma_j$. Then the expression $\sqrt{\sigma_i^2+\sigma_j^2}-(\sigma_i+\sigma_j)$ can be represented as a function of λ

$$\varphi(\lambda) = \sigma_i \cdot \left\lfloor \sqrt{1+\lambda^2} - (1+\lambda) \right\rfloor . \tag{15}$$

While $\lambda \to +\infty$ $\varphi(\lambda) \to -\sigma_i$, as $\lim\limits_{\lambda \to +\infty} \left\lfloor \sqrt{1+\lambda^2} - \lambda \right\rfloor = 0$. The plot of function $\varphi(\lambda)$ when $\sigma_i = 1$ is presented on Figure 3.

Thus, we arrive at a number of important conclusions about the properties of a stochastic cooperative game with a normally distributed individual utilities of players.

- If we follow the criterion of exceeding VaR total utility over the sum of VaR-s of individual utilities, then the player whose individual stochastic utility $\tilde{v}(j)$ has a large standard deviation makes a "greater contribution" to the value of the "effect of adding up of utilities" $v^+_{1-\alpha}(i \cup j) - (v_{1-\alpha}(i) + v_{1-\alpha}(j))$.

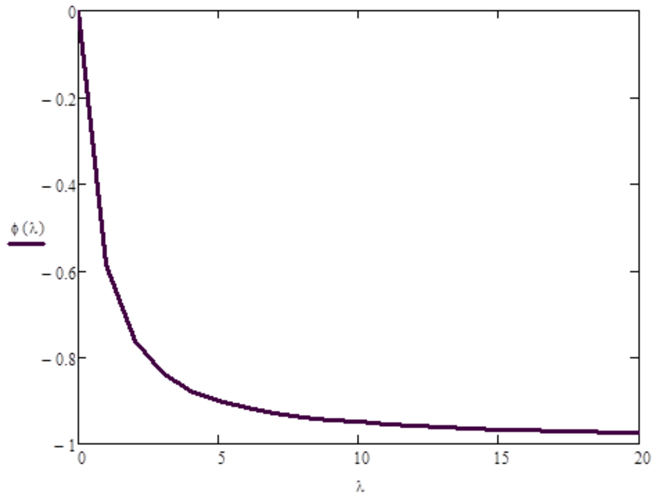


Figure3: Plot of function $\varphi(\lambda)$ while $\sigma_i = 1$

- With an increase in the variances of individual utilities the effect from adding them up will tend to decreasing of temps and approach the limit value $-\Phi^{-1}(1-\alpha) \cdot \sigma_i$, where $\sigma_i = \min\{\sigma_i, \sigma_j\}$.

It is easy to see that the proposed approaches can simply be extended to situations where we add up utilities of a random number of players ($S \subset I$)

$$\tilde{v}^{+}(S) = \sum_{i \in S} \tilde{v}(i).$$

The expression, which evaluates the effect of adding up our utilities takes the form

$$v^{+}_{1-\alpha}(S) - \sum_{i \in S} v_{1-\alpha}(i) = \Phi^{-1}(1-\alpha) \cdot \left[\sqrt{\sum_{i \in S} \sigma_i^2} - \sum_{i \in S} \sigma_i \right]. \tag{16}$$

It has to be admitted that much of the problems that arise in the study of stochastic cooperative games and superadditivity in the sense of definition (2), are more related to the probability theory than to the theory of games. At the same time it must be noted that nowadays there is a relatively small number of papers devoted to problems of quintile ratio of the sum of random variables and sums of quantiles. Among them, in particular, may be called the following papers (Liu and David , 1989; Watson and Gordon, 1986).

4. Imputations in stochastic cooperative games

When we try to answer the question of what is understood under a solution of a stochastic cooperative game, we realize that we need to define an idea of imputation for this class of games. It seems logical and natural in terms of approaches that we have applied earlier, to define it as a vector $x(\alpha) \in R^m$ satisfying the conditions of

(a) individual rationality

$$P\{x_i(\alpha) \geq \tilde{v}(i)\} \geq \alpha \quad (\text{or} \quad x_i(\alpha) \geq F^{-1}_{\tilde{v}(i)}(\alpha) = v_\alpha(i)), \tag{17}$$

(b) group rationality of players

$$P\{\sum_{i=1}^{m} x_i(\alpha) \leq \tilde{v}(I)\} \geq \alpha \quad (\text{or} \quad \sum_{i=1}^{m} x_i(\alpha) \leq F^{-1}_{\tilde{v}(I)}(\alpha) = v_\alpha(I)). \tag{18}$$

We should pay our attention to some fundamental features of the proposed definition. Condition (17) essentially means that the utility x_i which is received by the player i in accordance with the imputation $x(\alpha)$ has to be not less than his individual random utility with the probability not less than α.

Thus, according to the requirement of individual rationality an imputation should provide each user with a utility that won't be less than the VaR utility (for the chosen level of probability α).

Condition (18) is a generalization of conditions of a group rationality in classical cooperative games with transferable utility. As it is known, according to this condition an imputation should fully distribute full benefit of a full (or "big") coalition

$$\sum_{i=1}^{n} x_i = v(I).$$

Then the transformation of this requirement into a requirement, under which the total utility distributed by an imputation should not exceed the value of the random payoff of the grand coalition (with a given level of probability α) is logical for the case of stochastic games

$$P\{\sum_{i=1}^{m} x_i(\alpha) \leq \tilde{v}(I)\} \geq \alpha.$$

Third, it is logical and reasonable to introduce an imputation with respect to a certain level of probability α in a stochastic cooperative game. In other words, the vector $x(\alpha_1)$, which is an imputation for the level of probability α_1 may not be an imputation for the level $\alpha_2 > \alpha_1$ in a general case.

Finally, we can naturally expand approaches to defining solutions on stochastic cooperative games. In particular under a stochastic core C_α, we will understand a set of imputation:

$$\begin{aligned} C_\alpha(\tilde{v}) = \{x \in R^{|I|} \mid \forall S \neq \emptyset, I : & \; P\{\tilde{v}(S) \leq x(S)\} \geq \alpha; \\ & \; P\{\tilde{v}(I) \geq x(I)\} \geq \alpha\}. \end{aligned} \tag{19}$$

5. Conclusions

In conclusion, we would additionally emphasize that despite the abstract nature of stochastic cooperative games as mathematical objects, despite the need for a substantial simplification of the initial economic processes and facts during construction of models that correspond to this class of games, they have a relatively high application potential, in our opinion. In particular, if we apply such models to large-scale economic projects, they can explain us preferences of potential participants to create some types if coalitions and, on the contrary, to reject some of them in spite of the fact that they competing in terms of expected income.

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A Differential Game-Based Approach to Extraction of Exhaustible Resource with Random Terminal Instants

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Abstract We investigate a noncooperative differential game in which two firms compete in extracting a unique nonrenewable resource over time. The respective times of extraction are random and after the first firm finishes extraction, the remaining one continues and gets the final reward for winning. An example is introduced where the optimal feedback strategy, i.e. the optimal extraction rate, is calculated in a closed form.

Keywords: Differential game, exhaustible resources, random terminal time, Hamilton-Jacobi-Bellman equation

1. Introduction

In the last decades many economic models have been investigated with the precious help of the tools provided by differential game theory (see Dockner et al. (2000), Jørgensen and Zaccour (2007)). Both deterministic and stochastic approaches have been widely developed in a wide range of different frameworks.

The present paper aims to analyze a class of models of differential games where 2 firms are engaged in a competition of extraction of a nonrenewable resource. In particular, we consider a framework where the terminal instants of extraction are random variables having different cumulative distribution functions. The first firm which stops extracting is the loser, whereas the remaining firm gets a terminal reward and keeps extracting on its own until the exhaustion of the resource.

We are going to fully characterize the structure of the game and to determine its dynamic equilibrium structure. Finally, we will feature an example which is a modification of the standard model of extraction (see Rubio (2006)), with linear state dynamics and a logarithmic payoff structure. It will be completely discussed and its optimal feedback solution will be exhibited.

The rest of this paper is organized as follows: in Section 2, the basic characteristics of the class of games under consideration are introduced. In Section 3, the Hamilton-Jacobi-Bellman equations for the feedback information structure are determined, whereas in Section 4 an example is featured and its solution is computed in a closed form. Section 5 concludes the paper and outlines some possible future developments.

2. The Problem Statement

Consider 2 firms involved in a noncooperative differential game of resource extraction with the following setup:

- given the different characteristics of the 2 firms, each one of them has a distinct terminal time of extraction of the same resource;
- as soon as the first one finishes, it quits the game and there remains just one firm left, which keeps extracting until its terminal time;
- the payoff of the game is composed of two components: the integral payoff achieved while playing, and the final reward, assigned to the player which stays alive after the retirement of its rival;
- the control variables of the players are their respective extraction rates $u_1(t)$, $u_2(t) \in R_+$;
- the unique state variable of the game is the stock of resource $x(t) \in R_+$, whose evolutionary dynamics is expressed by the following differential equation:

$$\begin{cases} \dot{x}(t) = \phi(t, x, u_1, u_2) \\ x(0) = x_0 > 0 \end{cases}, \tag{1}$$

  where the transition function $\phi(\cdot) \in C^2(\mathbb{R}^4_+)$ is negatively affected by the firms' extraction efforts:

$$\frac{\partial \phi}{\partial u_i} \leq 0, \qquad \text{for } i = 1, 2;$$

- we denote by $h_i(t, x, u_1, u_2) \in C^2(\mathbb{R}^4_+)$ the utility function of the i-th firm. No intertemporal discount factor appears in the functional objectives of the problem, because the discount structure is built on the characteristics of the random terminal instants.

Let T_1 and T_2 be the random variables denoting the respective terminal instants of the extracting firms, and assume that their c.d.f. $F_1(\cdot)$, $F_2(\cdot)$ and their p.d.f. $f_1(\cdot)$ and $f_2(\cdot)$ are known.

We impose an asymmetry condition concerning the longevity of players: calling $\omega_i > 0$ the upper bound of T_i, it is not restrictive to posit $\omega_1 > \omega_2$. Hence, the two p.d.f. naturally differ:

$$F_1(t) < 1 \quad \forall\, t < \omega_1, \qquad F_1(\omega_1) = 1;$$

$$F_2(t) < 1 \quad \forall\, t < \omega_2, \qquad F_2(t) = 1 \quad \forall\, t \in [\omega_2,\ \omega_1].$$

At time $T = \min\{T_1, T_2\}$, if player i is the only one remaining in the extraction game, she receives the terminal payoff $\Phi_i(x(T))$, subsequently, since she keeps playing on her own, the game collapses to an optimal control problem.

If we indicate with x^*, u_1^*, u_2^* the optimal state and strategies, and with $h_i^*(t) = h_i(t, x^*, u_1^*, u_2^*)$, the expected payoff for the i-th player in the problem (1) will be written as follows:

$$K_i(0, x_0, u_1^*, u_2^*) = \mathbb{E}\left[\int_0^{T_i} h_i^*(t)dt I_{[T_i < T_j]} + \int_0^{T_j} h_i^*(t)dt I_{[T_i > T_j]} + \Phi_i(x^*(T))\mathbb{I}_{[T_i > T_j]}\right], \tag{2}$$

where $\mathbb{I}_{[\cdot]}$ is the indicator function and $\mathbb{E}[\cdot]$ is the mathematical expectation.

3. Hamilton-Jacobi-Bellman equations

From now on, we will write $\omega = \omega_1$ in order to simplify notation. If (2) exists and is finite, then it can be decomposed in the following sum of the expected payoff plus the expected reward:

$$\mathbb{E}\left[\int_0^{T_i} h_i^*(t)dtI_{[T_i<T_j]} + \int_0^{T_j} h_i^*(t)dtI_{[T_i>T_j]} + \Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]}\right] =$$

$$= \mathbb{E}\left[\int_0^{T_i} h_i^*(t)dtI_{[T_i<T_j]} + \int_0^{T_j} h_i^*(t)dtI_{[T_i>T_j]}\right] + \mathbb{E}\left[\Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]}\right]. \quad (3)$$

From now on, we will write the terms of (3) as follows:

$$\begin{cases} \Psi_1^i(T_1,T_2) := \int_0^{T_i} h_i^*(t)dtI_{[T_i<T_j]} + \int_0^{T_j} h_i^*(t)dtI_{[T_i>T_j]} \\ \\ \Psi_2^i(T_1,T_2) := \Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]} \end{cases}.$$

We are going to separately calculate the two related expected values in the next two Propositions.

Proposition 1.

$$\mathbb{E}\left[\Psi_1^i(T_1,T_2)\right] = \mathbb{E}\left[\int_0^{\min\{T_1,T_2\}} h_i^*(t)dt\right].$$

Proof. Since T_1 and T_2 are independent random variables, the p.d.f. of the random vector (T_1, T_2) must be the product of their p.d.f's, i.e. an expression of the kind $f_1(\theta)f_2(\tau)$. We can note that:

$$\mathbb{E}\left[\Psi_1^i(T_1,T_2)\right] = \int_0^{\omega}\int_0^{\omega}\int_0^{\theta} h_i^*(t)dtI_{[\theta<\tau]}f_2(\tau)d\tau f_1(\theta)d\theta +$$

$$+ \int_0^{\omega}\int_0^{\omega}\int_0^{\tau} h_i^*(t)dtI_{[\theta>\tau]}f_1(\theta)d\theta f_2(\tau)d\tau. \quad (4)$$

From now on, call $H_i(\theta) := \int_0^{\theta} h_i^*(t)dt$. Hence, (4) amounts to:

$$\int_0^{\omega}\left(\int_0^{\tau} H_i(\theta)f_1(\theta)d\theta\right)f_2(\tau)d\tau + \int_0^{\omega}\left(\int_0^{\theta} H_i(\tau)f_2(\tau)d\tau\right)f_1(\theta)d\theta. \quad (5)$$

Integrating by parts twice and taking into account that $F_1(\omega) = F_2(\omega) = 1$, we obtain that the sum (5) is:

$$\int_0^{\tau} H_i(\theta)f_1(\theta)d\theta F_2(\omega) - \int_0^{\omega} H_i(\theta)f_1(\theta)F_2(\theta)d\theta +$$

$$+ \int_0^{\theta} H_i(\tau)f_2(\tau)d\tau F_1(\omega) - \int_0^{\omega} H_i(\tau)f_2(\tau)F_1(\tau)d\tau =$$

$$= H_i(\omega)F_1(\omega) - \int_0^\omega h_i^*(\theta)F_1(\theta)d\theta - H_i(\omega)F_1(\omega)F_2(\omega)+$$

$$+\int_0^\omega F_1(\theta)[h_i^*(\theta)F_2(\theta) + H_i(\theta)f_2(\theta)]d\theta+$$

$$+H_i(\omega)F_2(\omega) - \int_0^\omega h_i^*(\tau)F_2(\tau)d\tau - H_i(\omega)F_1(\omega)F_2(\omega)+$$

$$+\int_0^\omega F_2(\tau)[h_i^*(\tau)F_1(\tau) + H_i(\tau)f_1(\tau))]d\tau =$$

$$= -\int_0^\omega h_i^*(\tau)[F_1(\tau) + F_2(\tau) - 2F_1(\tau)F_2(\tau)]d\tau + \int_0^\omega H_i(\theta)[F_1(\tau)F_2(\tau)]'d\tau =$$

$$= -\int_0^\omega h_i^*(\tau)[F_1(\tau) + F_2(\tau) - 2F_1(\tau)F_2(\tau)]d\tau+$$

$$+H_i(\omega)F_1(\omega)F_2(\omega) - \int_0^\omega h_i^*(\tau)[F_1(\tau)F_2(\tau)]d\tau =$$

(and since $H_i(\omega) = \int_0^\omega h_i^*(\tau)d\tau$)

$$= \int_0^\omega h_i^*(\tau)d\tau - \int_0^\omega h_i^*(\tau)[F_1(\tau) + F_2(\tau) - F_1(\tau)F_2(\tau)]d\tau =$$

$$= \int_0^\omega h_i^*(\tau)[1 - F_1(\tau) - F_2(\tau) + F_1(\tau)F_2(\tau)]d\tau =$$

$$= \int_0^\omega h_i^*(\tau)[1 - F_1(\tau)][1 - F_2(\tau)]d\tau = \int_0^\omega h_i^*(\tau)[1 - F(\tau)]d\tau,$$

where $F(\cdot)$ is the c.d.f. of the variable $T = \min\{T_1,\ T_2\}$, which completes the proof.

Proposition 2.

$$\mathbb{E}\left[\Psi_2^i(T_1, T_2)\right] = \int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))d\tau.$$

Proof. Integrating by parts and taking into account that $F_i(\omega) = 1$, we have that:

$$\mathbb{E}\left[\Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]}\right] = \int_0^\omega \left(\int_0^\omega \Phi_i(x^*(\tau))I_{[\theta>\tau]}f_j(\tau)d\tau\right) f_i(\theta)d\theta =$$

$$= F_i(\omega)\int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)d\tau - \int_0^\omega F_i(\theta)\Phi_i(x^*(\theta))f_j(\theta)d\theta =$$

$$= \int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)d\tau - \int_0^\omega F_i(\theta)\Phi_i(x^*(\theta))f_j(\theta)d\theta,$$

then, by considering a unique variable for integration, we conclude that

$$\mathbb{E}\left[\Phi_i(x^*(T))\mathbb{I}_{[T_i>T_j]}\right] = \int_0^\omega \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))d\tau.$$

Hence, Propositions 1 and 2 entail the following result:

Corollary 1. *The expected payoff (2) for the problem starting at $t = 0$ is given by:*

$$K_i(0, x_0, u_1^*, u_2^*) = \int_0^{\omega} h_i^*(\tau)[1 - F(\tau)] + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))d\tau. \quad (6)$$

Furthermore, if we consider any subgame starting at a subsequent instant $t > 0$, we have to take into account the possibility that such game may not start at all, namely that $0 < T < t$. The related conditional probability can be expressed by dividing the payoff integral by both the probabilities that $t < T_1$ and that $t < T_2$, i.e. $(1 - F_1(t))(1 - F_2(t)) = 1 - F(t)$.

The rationale for this is based on the fact that since we do not know the terminal time of the game a priori, the payoff can be only defined by ensuring that the initial instant is strictly smaller than both possible terminal instants.

We denote by $W_i(t, x)$ the i-th optimal value function of the problem starting at $t \in (0, \omega)$, with initial data $x(t) = x$. We have that:

$$W_i(t, x) =$$
$$\frac{1}{(1 - F_1(t))(1 - F_2(t))} \int_t^{\omega} [h_i^*(\tau)\,(1 - F(\tau)) + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))]\,d\tau. \quad (7)$$

If we call

$$\widetilde{W_i}(t, x) := \int_t^{\omega} [h_i^*(\tau)\,(1 - F(\tau)) + \Phi_i(x^*(\tau))f_j(\tau)(1 - F_i(\tau))]\,d\tau, \quad (8)$$

the relation $\widetilde{W_i}(\cdot) = (1 - F_1(t))(1 - F_2(t))W_i(\cdot)$ holds. Calculating the relevant first order partial derivatives of (8) yields:

$$\frac{\partial \widetilde{W_i}(t, x)}{\partial t} =$$
$$(1 - F_1(t))(1 - F_2(t))\frac{\partial W_i(t, x)}{\partial t} - W_i(t, x)\,[f_1(t)(1 - F_2(t)) + (1 - F_1(t))f_2(t)]\,,$$

$$\frac{\partial \widetilde{W_i}(t, x)}{\partial x} = (1 - F_1(t))(1 - F_2(t))\frac{\partial W_i(t, x)}{\partial x}.$$

Consequently, after renaming $\widetilde{W_i} := W_i$, the Hamilton-Jacobi-Bellman equations can be rewritten as follows:

$$-\frac{\partial \widetilde{W_i}(t, x)}{\partial t} = \max_{u_i}[h_i(t, x, u_1, u_2)(1 - F_1(t))(1 - F_2(t)) +$$
$$\Phi_i(x(t))f_j(t)(1 - F_i(t)) + \frac{\partial \widetilde{W_i}(t, x)}{\partial x}\phi(t, x, u_1, u_2)], \quad (9)$$

then, dividing both sides by $(1 - F_1(t))(1 - F_2(t))$, we obtain:

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\frac{f_1(t)}{1-F_1(t)} + \frac{f_2(t)}{1-F_2(t)}\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\frac{f_j(t)}{1-F_j(t)} + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \tag{10}$$

Finally, employing the form of the hazard functions $\lambda_i(t) := \dfrac{f_i(t)}{1-F_i(t)}$, the Hamilton-Jacobi-Bellman equations read as:

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\lambda_1(t) + \lambda_2(t)\right] =$$
$$\max_{u_i}[h_i(t,x,u_1,u_2) + \Phi_i(x(t))\lambda_j(t) + \frac{\partial W_i(t,x)}{\partial x}\phi(t,x,u_1,u_2)]. \tag{11}$$

4. An example

Consider the following framework, borrowed from Rubio (2006) (Example 2.1) and Dockner et al. (2000) (Example 5.7) and modified with the above discount factor. This example originally describes the joint exploitation of a pesticide, but its structure makes it suitable for our aim. Note that, in contrast to Rubio (2006), we confine our attention to the Nash equilibrium under simultaneous play, and we consider the non-stationary feedback case, that is our optimal value function explicitly depends on the initial instant t.

We fix $m = 1$, i.e., a unique state variable $x(t)$, denoting the amount of the resource, whereas the i-th payoff function explicitly depends on the rate of extraction of the i-th player but not on the state variable:

$$h_i(x(t), u_i(t)) = \ln u_i(t),$$

whereas the terminal payoff is given by

$$\Phi_i(x^*(T)) = c_i \ln(x(T_i)).$$

Note that $h_i(\cdot)$ is well-defined and concave for $u_i > 0$.

The transition function is linear and decreasing in the controls, so the dynamic constraint is:

$$\begin{cases} \dot{x} = -u_1 - u_2 \\ x(0) = x_0 > 0 \end{cases}.$$

The kinematic equation ensures that the terminal payoff is well-defined in that the resource cannot equal 0 in finite time.

Using the data of the above model, we obtain:

$$W_i(0,x_0) = \mathbb{E}\left[\int_0^{T_i} \ln u_i^* dt I_{[T_i<T_j]} + \int_0^{T_j} \ln u_i^* dt I_{[T_i>T_j]} + c_i \ln x(T_j)\mathbb{I}_{[T_i>T_j]}\right].$$

The i-th optimal value function of the problem starting at $t \in (0,\omega)$, and with initial condition $x(t) = x$, is given by:

$$W_i(t,x) = \frac{1}{(1-F_i(t))(1-F_j(t))} \int\limits_t^{\omega} \left[\ln u_i^*(\tau, x(\tau))\,(1-F(\tau)) + c_i \ln x(\tau) f_j(\tau)(1-F_i(\tau))\right] d\tau. \tag{12}$$

In compliance with the previous Section, the Hamilton-Jacobi-Bellman equations are given by:

$$-\frac{\partial W_i(t,x)}{\partial t} + W_i(t,x)\left[\lambda_i(t)+\lambda_j(t)\right] = \max_{u_i}\left[\ln(u_i) + c_i \ln x(t)\lambda_j(t) - \frac{\partial W_i(t,x)}{\partial x}(u_i+u_j^*)\right]. \tag{13}$$

In order to explicitly determine the optimal strategy in the feedback Nash structure, we guess the following ansatz for the solution to (13):

$$W_i(t,x) = A_i(t)\ln x + B_i(t),$$

where $A_i(t)$ and $B_i(t)$ are unknown functions of t, such that the following limits are satisfied:

$$\lim_{t\to\omega} A_i(t) = 0, \qquad \lim_{t\to\omega} B_i(t) = 0. \tag{14}$$

The relevant first order partial derivatives to be employed in (13) are:

$$\frac{\partial W_i(t,x)}{\partial t} = \dot{A}_i(t)\ln x + \dot{B}_i(t), \qquad \frac{\partial W_i(t,x)}{\partial x} = \frac{A_i(t)}{x}.$$

Maximizing the r.h.s. of (13) yields:

$$\frac{1}{u_i^*} - \frac{\partial W_i(t,x)}{\partial x} = 0 \iff u_i^* = \frac{x}{A_i(t)}.$$

Hence, plugging u_i^*, $\frac{\partial W_i(t,x)}{\partial t}$ and $\frac{\partial W_i(t,x)}{\partial x}$ into (13), we obtain the following equation:

$$-\dot{A}_i(t)\ln x - \dot{B}_i(t) + (A_i(t)\ln x + B_i(t))\left[\lambda_i(t)+\lambda_j(t)\right] = \ln\frac{x}{A_i(t)} + c_i \ln x \lambda_j(t) - \frac{A_i(t)}{x}\left(\frac{x}{A_i(t)} + \frac{x}{A_j(t)}\right). \tag{15}$$

After collecting terms with and without $\ln x$, we determine the following ODEs for the time-dependent coefficients of $W_i(t,x)$:

$$-\dot{A}_i(t) + A_i(t)\left[\lambda_i(t)+\lambda_j(t)\right] - 1 - c_i\lambda_j(t) = 0, \tag{16}$$

$$-\dot{B}_i(t) + B_i(t)\left[\lambda_i(t)+\lambda_j(t)\right] + \ln A_i(t) + 1 + \frac{A_i(t)}{A_j(t)} = 0, \tag{17}$$

composing a Cauchy problem endowed with the transversality conditions:

$$\lim_{t\longrightarrow\omega} A_i(t) = 0, \qquad \lim_{t\longrightarrow\omega} B_i(t) = 0. \tag{18}$$

Proposition 3. *The optimal feedback strategy for the i-th firm is given by:*

$$u_i^*(t,x) = \frac{x}{\int_t^{\omega}(1+c_i\lambda_j(\tau))e^{-\int_t^{\tau}(\lambda_i(\theta)+\lambda_j(\theta))d\theta}d\tau}. \tag{19}$$

Proof. We just consider the Cauchy problem in $A_i(t)$, because the explicit calculation of $B_i(t)$ can be avoided in that $B_i(t)$ does not appear in the expression of u_i^*:

$$\begin{cases} \dot{A}_i(t) = A_i(t)\left[\lambda_i(t)+\lambda_j(t)\right] - 1 - c_i\lambda_j(t) \\ \lim\limits_{t\to\omega} A_i(t) = 0 \end{cases},$$

whose general solution is given by:

$$A_i(t) = e^{\int_0^t(\lambda_i(\tau)+\lambda_j(\tau))d\tau}\left(C - \int_0^t (1+c_i\lambda_j(\tau))e^{-\int_0^{\tau}(\lambda_i(s)+\lambda_j(s))ds}d\tau\right), \tag{20}$$

where the constant C is determined by employing the transversality condition on $A_i(t)$:

$$C = \int_0^{\omega}(1+c_i\lambda_j(\tau))e^{-\int_0^{\tau}(\lambda_i(s)+\lambda_j(s))ds}d\tau,$$

leading to the solution:

$$A_i^*(t) = e^{\int_0^t(\lambda_i(\tau)+\lambda_j(\tau))d\tau}\left[\int_t^{\omega}(1+c_i\lambda_j(\tau))e^{-\int_0^{\tau}(\lambda_i(s)+\lambda_j(s))ds}d\tau\right]. \tag{21}$$

We can simplify:

$$A_i^*(t) = \int_t^{\omega}(1+c_i\lambda_j(\tau))e^{-\int_t^{\tau}(\lambda_i(s)+\lambda_j(s))ds}d\tau. \tag{22}$$

Finally, the expression of the optimal feedback strategy for the i-th firm can be achieved from the FOCs of the model:

$$u_i^*(t,x) = \frac{x}{A_i^*(t)} = \frac{x}{\int_t^{\omega}(1+c_i\lambda_j(\tau))e^{-\int_t^{\tau}(\lambda_i(\theta)+\lambda_j(\theta))d\theta}d\tau}. \tag{23}$$

As a further application, we can consider the circumstance where the two distributions of the firms are the standard exponential distributions, i.e.

$$f_i(t;\ \lambda_i) = \begin{cases} \lambda_i e^{-\lambda_i t}, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases},$$

whose means are respectively λ_1^{-1}, λ_2^{-1}, both positive, with $\lambda_1 \neq \lambda_2$, ensuring asymmetry.

In this case the hazard functions are constant, i.e. $\lambda_1(t) \equiv \lambda_1$ and $\lambda_2 \equiv \lambda_2$, then substituting in (19) we obtain the two optimal feedback strategies:

$$u_1^*(t,x) = \frac{(\lambda_1+\lambda_2)x}{(1+c_1\lambda_2)[1-e^{-(\lambda_1+\lambda_2)(\omega-t)}]}, \tag{24}$$

$$u_2^*(t,x) = \frac{(\lambda_1+\lambda_2)x}{(1+c_2\lambda_1)[1-e^{-(\lambda_1+\lambda_2)(\omega-t)}]}. \tag{25}$$

5. Concluding remarks

This paper intends to be a contribution to the literature of differential games in an area which can be defined as deterministic, but enriched with some stochastic elements. In particular, it is focused on the feature of extraction games that is definitely realistic: the uncertainty about the terminal times of an extracting activity.

The dynamic feedback equilibrium structure has been determined and the specific technicalities of this setting have been pointed out. As an example, a model of nonrenewable resource extraction with a logarithmic utility structure was examined and solved in a closed form.

There exist some possible further extensions, also concerning the example we developed. It would be interesting to check the specific optimal strategies in presence of more complex hazard functions (for example, the Weibull distribution) or endowed with alternative payoff structures. Another interesting development might consist in considering a competition among more than 2 firms, having different terminal times.

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On the Consistency of Weak Equilibria in Multicriteria Extensive Games

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Abstract This paper considers weak equilibria properties for multicriteria n-person extensive games. It is shown that the set of subgame perfect weak equilibriums in multicriteria games with perfect information is non-empty, however one can not use the backwards induction procedure (in the direct way) to construct equilibria in multicriteria extensive game.
Furthermore, we prove that weak equilibria satisfies time consistency in multicriteria extensive games (with perfect or incomplete information).

Keywords: multicriteria games, extensive games, equilibria, time consistency

1. Introduction

We deal with so-called multicriteria games (or the games with vector payoffs) when every player takes several criteria into account. Shapley (1959) defined the notion of equilibrium point for (two-person) games with vector-payoffs and showed the correspondence between equilibria and Nash equilibria (Nash, 1950) of so-called trade-off games.

The basic results for Nash equilibria in n-person extensive games with incomplete information were elaborated in (Kuhn, 1953). The main results concerning time consistency of optimality principles in extensive games were summarized in (Petrosjan and Kuzyutin, 2008).

Some interesting properties of equilibria in different classes of multicriteria extensive games were established in (Borm, 1999), (Petrosjan and Puerto, 2002), (Fahretdinova, 2002), (Kuzyutin and Nikitina, 2011).

The main purpose of this paper is to extend some results concerning Nash equilibria in n-person extensive unicriterium games to weak equilibria in multicriteria extensive games (with perfect and incomplete information).

Section 2 contains main notations used in extensive games. Section 3 contains brief summary on decomposition of extensive games and strategies.

The example in section 4 shows one undesireable property of weak equilibria in multicriteria extensive games: if we have equilibrium φ^x in the subgame and equilibrium φ^D in corresponding factor-game, the "composite behavior" $\varphi = (\varphi_i^D, \varphi_i^x)_{i=1}^n$ does not necessarily satisfy the equilibria condition in the original extensive game.

The existence theorem (for weak equilibrium in pure strategies in multicriteria extensive games with perfect information) is proved in section 5. A slight modification of backwards induction procedure (which allows to construct subgame perfect weak equilibria) is also presented in this section.

The time consistency of weak equilibria in multicriteria extensive games (with perfect or incomplete information) is proved in sections 6 an 7.

2. Multicriteria n-person extensive games with perfect information

We'll use the following notations (Kuhn, 1953; Petrosjan and Kuzyutin, 2008):

- $\Gamma = \{N, K, P, A, h\}$ — finite multicriteria n-person game (or the game with vector payoffs) in extensive form with perfect information;
- $N = \{1, \ldots, n\}$ — the set of players in Γ;
- K — the game tree (with initial node x_0) that consists of the set Z of all terminal nodes (endpoints) and the set $X = K \setminus Z$ of all intermediate nodes;
- $x < y$ means that (unique) path from x_0 to y contains x, and $x \neq y$;
- $S(x)$ — the set of all node x immediate "successors"; $S(x) = \emptyset \quad \forall\, x \in Z$;
- $S^{-1}(x)$ — the unique immediate "precessor" of the node x: $x \in S\left(S^{-1}(x)\right)$, $S^{-1}(x_0) = \emptyset$;
- $Z(x)$ — the set $\{y \in Z \mid x < y\}$, i.e. the set of terminal nodes, which can be reached from x;
- $\omega = \{x_0, x_1, x_2, \ldots, x_l\}$ — the play (or trajectory) of length l:

$$x_0 < x_1 < \ldots < x_l, \quad x_l \in Z,$$

$$x_{j-1} = S^{-1}(x_j), \quad j = 1, \ldots, l.$$

- P_i — is the set of all nodes where player i moves,

$$\bigcup_{i \in N} P_i = K \backslash Z;$$

- A — the "choice partition", i.e.:

$$A_j = \{x \in K \setminus Z \mid |S(x)| = j\};$$

- $h_i(z) = (h_{i/1}(z), \ldots, h_{i/r(i)}(z))$ — the player i payoffs vector at the terminal node $z \in Z$.

The player's i pure strategy is a function (with domain P_i) that determines for every node $x \in P_i$ some choice or alternative $y \in S(x)$.

The set of all player's i pure strategies in Γ denote by $\Phi_i, i \in N$. The strategy profile $\varphi = (\varphi_1, \ldots, \varphi_n)$ determines a unique play $\omega = \{x_0, x_1, \ldots, x_l\}$ in Γ, where $\varphi_i(x_k) = x_{k+1}$, if $x_k \in P_i, x_l \in Z$, and, correspondingly, a collection of all players vector payoffs $\{h_i(x_l)\}_{i \in N}$.

Due to one-one mapping between the all plays ω set and the set Z of all terminal nodes, we'll use the following notation:

$$h_i(\omega) = h_i(x_l), \text{where } \omega = \{x_0, x_1, ..., x_l\}, x_l \in Z.$$

Denote by H_i the $r(i)$-vector valued payoff function, that assigns to each strategy profile $\varphi = (\varphi_1, ..., \varphi_n)$ the corresponding player i vector payoff:

$$H_i : \prod_{j=1}^{n} \Phi_j \longrightarrow R^{r(i)} \tag{1}$$

Note that player i in multicriteria game Γ tries to maximize $r(i)$ scalar criteria (i.e. all the components of his vector valued payoff function $H_i(\varphi) = (H_{i|1}(\varphi), ..., H_{i|r(i)}(\varphi))$).

Denote by $MG^p(n, K, r(1), \dots, r(n))$ the class of all finite n-person multicriteria extensive games with perfect information and vector payoffs (1).

3. The decomposition of extensive games and strategies

In a game Γ with perfect information every intermediate node $x \in K\backslash Z$ generates the subgame $\Gamma_x = \{N^x, K^x, P^x, A^x, h^x\}$, which components are just the restrictions (Kuhn, 1953; Petrosjan and Kuzyutin, 2008) of corresponding components of the original game Γ onto subtree K_x (the subgame Γ_x tree).

In particular,

$$h_i^x(y) = h_i(y) \quad \forall y \in Z(x) \quad \forall i \in N \tag{2}$$

Denote by Φ_i^x the set of all player's i pure strategies in the subgame Γ_x. The strategy profile $\varphi^x \in \prod_{j=1}^{n} \Phi_i^x$ generates the unique play $\omega^x = \{x, \dots, x_m\}$ in the subgame and, hence, the collection of players' vector payoffs:

$$H_i^x : \prod_{j=1}^{n} \Phi_j^x \longrightarrow R^{r(i)}, i \in N. \tag{3}$$

Let $x \in K\backslash Z, x \neq x_0$. For every strategy profile φ^x in the subgame Γ_x denote by $\Gamma_D = \Gamma_D(\varphi^x)$ the so-called factor-game on the tree $K^D = \{x\} \cup K\backslash K^x$.

Note that $\{x\} \cup Z\backslash Z(x)$ — the set of terminal nodes in factor-game, and

$$h_i^D(x) = H_i^x(\varphi^x), i \in N. \tag{4}$$

Denote by Φ_i^D the set of all player's i pure strategies in factor-game Γ_D. The strategy profile $\varphi^D \in \prod_{j=1}^{n} \Phi_i^D$ generates the unique play $\omega^D = \{x_0, \dots, x_k\}$ in the factor-game Γ_D and, hence, the collection of players' vector payoffs:

$$H_i^D : \prod_{j=1}^{n} \Phi_j^D \longrightarrow R^{r(i)}, i \in N. \tag{5}$$

The decomposition of original extensive game Γ at the node x onto subgame Γ_x and factor-game Γ_D generates the corresponding decomposition of pure (and mixed) strategies (Kuhn, 1953; Petrosjan and Kuzyutin, 2008). The pure strategy $\varphi_i \in \Phi_i$ decomposition at intermediate node x onto pure strategy $\varphi_i^x \in \Phi_i^x$ in the subgame Γ_x and pure strategy $\varphi_i^D \in \Phi_i^D$ in the factor-game Γ_D means that:

- φ_i^x is the restriction of φ_i onto the set P_i^x;
- φ_i^D is the restriction of φ_i onto the set P_i^D of all player's i nodes in the factor-game Γ_D.

Note that $P_i = P_i^x \bigcup P_i^D$, and, hence, one can compose the player's pure strategy $\varphi_i = (\varphi_i^D, \varphi_i^x) \in \Phi_i$ in the original game Γ from his strategies $\varphi_i^x \in \Phi_i^x$ and $\varphi_i^D \in \Phi_i^D$ in the subgame Γ_x and factor-game Γ_D correspondly.

4. Subgame perfect weak equilibrium in multicriteria extensive game

Let $x, y \in R^t$, and $y > x$ means that $y_i > x_i$ for all $i = 1, \ldots, t$. The vector $x \in M \subseteq R^t$ is weak Pareto efficient (or undominated) in M if $\{y \in R^t \mid y > x\} \bigcap M = \emptyset$. In this case we'll use the following notation: $x \in WPO(M)$.

Given strategy profile $\widehat{\varphi} = (\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n) = (\widehat{\varphi}_i, \widehat{\varphi}_{-i})$ in the finite n-person extensive multicriteria game with perfect information $\Gamma \in MG^P(n, K, r(1), \ldots, r(n))$ denote by

$$M_i(\Gamma, \widehat{\varphi}_{-i}) = \{H_i(\varphi_i, \widehat{\varphi}_{-i}), \varphi_i \in \Phi_i\} \tag{6}$$

the set of all player's i attainable vector payoffs (due to arbitrary choice of his strategy $\varphi_i \in \Phi_i$).

Definition 1. The strategy profile $\widehat{\varphi} = (\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n)$ is called (weak) equilibrium (Borm, 1999) in multicriteria game $\Gamma \in MG^P(n, K, r(1), \ldots, r(n))$, iff

$$H_i(\widehat{\varphi}_i, \widehat{\varphi}_{-i}) \in WPO(M_i(\Gamma, \widehat{\varphi}_{-i})) \ \ \forall i \in N. \tag{7}$$

We let $ME(\Gamma)$ denote the set of all weak equilibriums in Γ. Note that (7) is equivalent to the following condition:

$$(\widehat{\varphi}_1, \ldots, \widehat{\varphi}_n) \in ME(\Gamma) \Longleftrightarrow \forall i \in N \ \rceil\exists \ \varphi_i \in \Phi_i : H_i(\varphi_i, \widehat{\varphi}_{-i}) > H_i(\widehat{\varphi}_i, \widehat{\varphi}_{-i}). \tag{8}$$

Definition 2. The strategy profile $\widehat{\varphi} \in ME(\Gamma)$ is called subgame perfect weak equilibrium in Γ iff:

$$\widehat{\varphi}^x \in ME(\Gamma^x) \ \ \forall x \in K \backslash Z. \tag{9}$$

Denote by $SPME(\Gamma)$ the set of all subgame perfect weak equilibriums in Γ.

One should note that in case $r(i) = 1$ for all $i \in N$ condition (8) coincides with usual Nash equilibria requirement (Nash, 1950) in unicriterium game.

Let us remember now the important result, established in (Kuhn, 1953) (using the decomposition of extensive games and players' strategies): if we have Nash equilibrium φ^x in the subgame Γ_x and Nash equilibrium φ^D in the corresponding factor-game $\Gamma_D(\varphi^x)$, the "the composite behavior" $\varphi = \{(\varphi_i^D, \varphi_i^x)\}_{i=1}^n$ forms the Nash equilibrium in original game Γ.

This basic result is valid not only for the games with perfect information (and pure strategies) but for the games with incomplete information as well (when players use mixed strategies in general case). More precisely, the following theorem holds (Kuhn, 1953; Petrosjan and Kuzyutin, 2008).

Theorem 1. *Let Γ be n-person extensive (unicriterium) game (with perfect or incomplete information), x — some intermediate node; $\bar{\varphi}^x = (\bar{\varphi}_1^x, ..., \bar{\varphi}_n^x)$ — the Nash equilibrium (in mixed strategies in general case) in the subgame Γ_x; $\bar{\varphi}^D = (\bar{\varphi}_1^D, ..., \bar{\varphi}_n^D)$ — the Nash equilibrium in factor-game $\Gamma_D(\bar{\varphi}^x)$.*

If every player's i strategy $\bar{\varphi}_i$ allows the decomposition onto $\bar{\varphi}_i^x$ and $\bar{\varphi}_i^D$ in the subgame Γ_x and factor-game Γ_D correspondly, then the strategy profile $\bar{\varphi} = (\bar{\varphi}_1, ..., \bar{\varphi}_n)$ forms the Nash equilibrium in the original game Γ.

This fact, in particular, allows to use the backwards induction procedure to construct subgame perfect equilibrium in unicriterium multistage game with perfect information (Petrosjan and Kuzyutin, 2008).

However, the following example shows that the same conclusion is not valid for weak equilibrium in multicriteria extensive games.

Example 1. Consider the multicriteria 2-person game with perfect information $\Gamma = (2, K, r(1) = 2, r(2) = 2)$ with game tree K, presented in fig. 1.

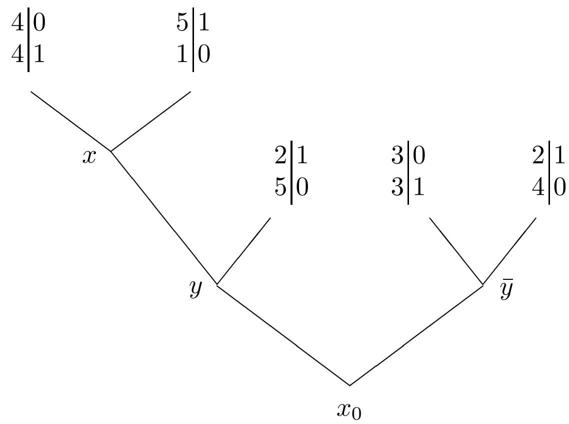

Figure1: 2-person multicriteria game Γ.

The players' vector payoffs are signed near every endpoint, $P_1 = \{x_0, x\}, P_2 = \{y, \bar{y}\}$.

The players' strategies $\varphi_1^y(x) = R$ (Right alternative at the node x) and $\varphi_2^y(y) = L$ form weak equilibrium in the subgame Γ_y:

$$\varphi^y = (\varphi_1^y, \varphi_2^y) \in ME(\Gamma_y).$$

Note, that in the factor-game $\Gamma_D(\varphi^y)$ the node y is terminal node with players' vector payoffs $\begin{array}{c|c} 5 & 1 \\ 1 & 0 \end{array}$.

It is also clear that the strategy profile $\varphi_1^D(x_0) = R$ and $\varphi_2^D(\bar{y}) = L$ is weak equilibrium in factor-game $\Gamma_D(\varphi)^y$:

$$\varphi^D = (\varphi_1^D, \varphi_2^D) \in ME(\Gamma_D(\varphi^y)),$$

and $H_1^D(\varphi^D) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

However, the "composite" strategy profile $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i = (\varphi_i^D, \varphi_i^x), i = 1, 2$, does not satisfy the equilibrium requirement (8), because

$$H_1(\varphi_1, \varphi_2) \notin WPO(M_1(\Gamma, \varphi_2)).$$

Indeed, consider the first player strategy $\psi_1(x_0) = L, \psi_1(x) = L$. Then

$$H(\psi_1, \varphi_2) = \begin{pmatrix} 4 \\ 4 \end{pmatrix} > H(\varphi_1, \varphi_2) = \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

5. The construction of SPME in multicriteria game

Unfortunately, one can not use the backwards induction procedure (in the direct way) to construct subgame perfect weak equilibriums in multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$.

To prove that the set $ME(\Gamma), \Gamma \in MG^P(n, K, r(1), ..., r(n))$, is nonempty let us consider auxiliary unicriterium game Γ_T. The only difference between original multicriteria game Γ and Γ_T is that every player in Γ_T tries to maximize only first criteria in his original vector payoff function. Thus, the player i payoff function in Γ_T is

$$H_i^T(\varphi) = H_{i|1}(\varphi) \quad \forall i \in N \quad \forall (\varphi_1, ..., \varphi_n) \in \prod_{j=1}^{n} \Phi_j. \tag{10}$$

Note that Γ_T is the usual (unicriterium) n-person extensive game with perfect information.

Lemma 1. *Let $\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n)$ — Nash equilibrium (in pure strategies) in unicriterium extensive game Γ_T with payoff function (10), that corresponds to multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$. Then*

$$\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n) \in ME(\Gamma).$$

Proof. By the NE requirement we have

$$H_{i|1}(\varphi_i, \hat{\varphi}_{-i}) \leq H_{i|1}(\hat{\varphi}_i, \hat{\varphi}_{-i}), \forall i \in N, \forall \varphi_i \in \Phi_i.$$

Hence,

$$H_{i|1}(\hat{\varphi}_i, \hat{\varphi}_{-i}) = \max_{\varphi_i \in \Phi_i} H_{i|1}(\varphi_i, \hat{\varphi}_{-i}).$$

Thus, there is no such strategy $\varphi_i \in \Phi_i$ that the following strict inequality holds

$$H_i(\varphi_i, \hat{\varphi}_{-i}) > H_i(\hat{\varphi}_i, \hat{\varphi}_{-i}).$$

In that case the strategy profile $\hat{\varphi}$ obviously satisfies the ME requirement (8). Hence, $\hat{\varphi} \in ME(\Gamma)$. □

Lemma 2. *If $\hat{\varphi} = (\hat{\varphi}_1, ..., \hat{\varphi}_n)$ is subgame perfect equilibrium in Γ_T with payoff function (10), then*

$$\hat{\varphi} \in SPME(\Gamma).$$

Using lemma 1 and 2 and the fact that every finite n-person extensive game with perfect information possesses SPE (in pure strategies) we get the following result.

Theorem 2. *Every finite n-person extensive multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information possesses subgame perfect weak equilibrium $\hat{\varphi} \in SPME(\Gamma)$ in pure strategies.*

Corollary 1. *The set $ME(\Gamma)$ of all weak equilibriums (in pure strategies) in finite n-person multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information is non-empty.*

To construct the set $MSPE(\Gamma)$ in finite n-person multicriteria extensive game Γ with perfect information one can use another auxiliary unicriterium game (so called "trade-off unicriterium game"), suggested in (Shapley, 1959). Let

$$\lambda(i) \in \Lambda_{r(i)} = \{\lambda \in R^{r(i)} | \lambda_j \geq 0, \lambda_1 + ... + \lambda_{r(i)} = 1\}$$

denote the player i "trade-off vector", and

$$H_i^{\lambda}(\varphi) = \sum_{j=1}^{r(i)} \lambda_j(i) \cdot H_{i|j}(\varphi). \tag{11}$$

denote the payoff function of player i in auxiliary unicriterium trade-off game Γ_λ.

Note that Γ_T is a partial case of trade-off game Γ_λ, when $\lambda_1(i) = 1, \lambda_j(i) = 0, j \neq 1$.

Let $NE(\Gamma_\lambda)$ denote the set of all Nash equilibriums in the trade-off game Γ_λ.

It was proved in (Shapley, 1959) that the set $ME(\Gamma)$ of all weak equilibriums in n-person multicriteria game Γ coincides with the set $NE(\Gamma_\lambda)$ of all Nash equilibriums in all auxiliary trade-off games Γ_λ i.e.

$$ME(\Gamma) = \{\hat{\varphi} \in NE(\Gamma_\lambda) | \lambda = (\lambda(1), ..., \lambda(n)) \in \Pi_{i=1}^n \Lambda_{r(i)}\}.$$

Using this basic result and lemma 1 and 2 we can propose the following technique to construct the set $ME(\Gamma)$ of all weak equilibriums (in pure strategies) in finite n-person multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information:

1) for every player $i \in N$ choose an arbitrary trade-off vector $\lambda(i) \in \Lambda_{r(i)}$.

2) apply the backwards induction procedure to auxiliary unicriterium trade-off game Γ_λ with payoff functions (11) to construct all subgame perfect equilibriums $\hat{\varphi} \in SPE(\Gamma_\lambda)$ in pure strategies. All these strategy profiles $\hat{\varphi}$ are subgame perfect weak equilibriums in the original multicriteria game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$.

6. Time consistency of week equilibria in multicriteria extensive games with perfect information

The strategy profile $\hat{\varphi} \in ME(\Gamma)$ generates the unique play (trajectory) ω on the game tree K in multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information. Let $G(\hat{\varphi})$ denote the set of all subgames along the play ω, i.e. $G(\hat{\varphi}) = \{\Gamma_x | x \in \omega\}$.

Definition 3. The set $ME(\Gamma)$ (the optimality principle ME) satisfies the time consistency property (Petrosjan and Kuzyutin, 2008) if for every weak equilibrium $\hat{\varphi} \in ME(\Gamma)$ and every subgame $\Gamma_x \in G(\hat{\varphi})$ the following inclusion holds: $\hat{\varphi}^x \in ME(\Gamma_x)$.

Theorem 3. *The set $ME(\Gamma)$ of all weak equilibriums in pure strategies in n-person multicriteria extensive game $\Gamma \in MG^P(n, K, r(1), ..., r(n))$ with perfect information satisfies the time consistency property.*

Proof. Let $\hat{\varphi} \in ME(\Gamma)$, i.e. condition (8) holds. Suppose that $\hat{\varphi}^x \notin ME(\Gamma)$ in some subgame $\Gamma_x \in G(\hat{\varphi})$. Then there exists such strategy $\varphi_i^x \in \Phi_i^x$ of some player i that

$$H_i^x(\varphi_i^x, \hat{\varphi}_{-i}^x) > H_i^x(\hat{\varphi}^x) = H_i(\hat{\varphi}).$$

At the same time

$$H_i^x(\varphi_i^x, \hat{\varphi}_{-i}^x) = H_i(\psi_i, \hat{\varphi}_{-i}), \text{ where } \psi_i = (\hat{\varphi}_i^D, \varphi_i^x) \in \Phi_i.$$

Hence we constructed such strategy ψ_i of player $i \in N$ in the original game Γ, that

$$H_i(\psi_i, \hat{\varphi}_{-i}) > H_i(\hat{\varphi}_i, \hat{\varphi}_{-i}).$$

However, the last vector inequality contradicts (8).

Hence, the set $ME(\Gamma), \Gamma \in MG^P(n, K, r(1), ..., r(n))$ is time consistent. $\square$

7. Weak equilibria in mixed strategies in multicriteria extensive games with incomplete information

Now let us consider the class $MG(n, K, r(1), ..., r(n))$ of finite n-person extensive games $\Gamma = \{N, K, P, A, U, h\}$ with incomplete information (Kuhn, 1953) and with vector payoffs. We let U denote the collection of all players informational sets. Note that the mixed strategy profile μ in extensive game $\Gamma = \{N, K, P, A, U, h\}$ with incomplete information generates in general case the whole set $\Omega(\mu)$ of plays (trajectories) ω on the game tree K, and let $p(\omega, \mu)$ denotes the probability of the play ω realization in Γ if all players use the mixed strategies $\mu_i, i \in N$.

Note, that the intermediate node x generates the subgame Γ_x (subgame on the tree K_x) of the game Γ with incomplete information iff every informational set in Γ is included in K_x or does not intersect with K_x.

Decomposition of extensive game Γ with incomplete information at the node x onto factor-game Γ_D and subgame Γ_x generates corresponding decomposition of mixed strategies (Kuhn, 1953; Petrosjan and Kuzyutin, 2008). In that case the following proposition holds.

Lemma 3. *Every pair μ_i^x and μ_i^D of player's i mixed strategies in Γ_x and Γ_D can be obtained as the result of decomposition of some mixed strategy μ_i in the original game Γ. Moreover, for each play $\omega \in \Gamma$ which contains x, the following condition holds:*

$$p(\omega, \mu) = p(\bar{\omega}_x, \mu^D) \cdot p(\omega^x, \mu^x), \tag{12}$$

where $\mu^D = (\mu_1^D, ..., \mu_n^D)$ — the strategy profile in Γ_D, $\mu^x = (\mu_1^x, ..., \mu_n^x)$ — the strategy profile in the subgame Γ_x, $\omega = \{x_0, ..., x, ..., x_l\}, x_l \in Z$ — the play (trajectory) in Γ, $\bar{\omega}_x = \{x_0, ..., x\}$ — the play in Γ_D, $\omega^x = \{x, ..., x_l\}$ — the play in Γ_x, $p(\bar{\omega}_x, \mu^D) = p(x, \mu^D)$ — the probability of reaching the node x if all players use the mixed strategies $\mu_i^D, i \in N$ in factor-game Γ^D.

As it was proved in (Fahretdinova, 2002), the set $SPME(\Gamma)$ of all subgame perfect weak equilibriums (in mixed strategies)in finite n-person extensive multicriteria game with incomplete information is non-empty.

Moreover, note that one can apply the technique for $SPME$ construction (which we suggested in section 5) in multicriteria extensive games with incomplete information as well.

Let $\hat{\mu} \in ME(\Gamma), \Gamma \in MG(n, K, r(1), ..., r(n))$, generates the set $\Omega(\hat{\mu})$ of optimal plays ω on the game tree K and $G(\hat{\mu})$ — the set of all possible subgames Γ_x along the "optimal game evolution", i.e. $x \in \omega, \omega \in \Omega(\hat{\mu})$.

Theorem 4. *The set $ME(\Gamma)$ of all weak equilibriums (in mixed strategies) in the game $\Gamma \in MG(n, K, r(1), ..., r(n))$ with incomplete information satisfies the time consistency property.*

Proof. $\hat{\mu} \in ME(\Gamma)$ iff every player i has no such mixed strategy μ_i that:

$$H_i(\mu_i, \hat{\mu}_{-i}) > H_i(\hat{\mu}_i, \hat{\mu}_{-i}). \tag{13}$$

Let $\Gamma_x \in G(\hat{\mu})$, i.e. $x \in \omega_n, \omega_n \in \Omega(\hat{\mu}), x \neq x_0$. Note that the set of all optimal trajectories $\{\omega_n\}$, generated by $\hat{\mu}$ can be divided onto two subsets: $\{\eta_m\} = \{\omega | x \in \omega\}$ and $\{\chi_k\} = \{\omega | \omega \text{ does not contain } x\}$, and $\{\eta_m\} \bigcap \{\chi_k\} = \emptyset$.

Then

$$H_i(\hat{\mu}) = \sum_m p(\eta_m, \hat{\mu}) \cdot h_i(\eta_m) + \sum_k p(\chi_k, \hat{\mu}) \cdot h_i(\chi_k) \tag{14}$$

Let $\hat{\mu}^D = (\hat{\mu}_1^D, ..., \hat{\mu}_n^D)$ — the result of strategy profile μ decomposition, corresponding to factor-game $\Gamma_D = \Gamma_D(\hat{\mu}^x)$, and

$$p(\bar{\eta}_x, \hat{\mu}^D) = p(x, \hat{\mu}^D) = p(x, \hat{\mu})$$

— the probability of reaching the node x (or the probability of play $\bar{\eta}_x = \{x_0, ..., x\}$) in factor-game Γ_D, when all players use strategies $\hat{\mu}_i^D, i \in N$.

Suppose that the time consistency condition is violated in the subgame Γ_x, i.e. $\hat{\mu}^x \notin ME(\Gamma_x)$. Then for some player $i \in N$ there exists such strategy μ_i^x in Γ_x that

$$H_i^x(\mu_i^x, \hat{\mu}_{-i}^x) > H_i^x(\hat{\mu}_i^x, \hat{\mu}_{-i}^x) \tag{15}$$

Let the strategy profile $(\mu_i^x, \hat{\mu}_{-i}^x)$ generates the set of plays $\{\xi_\alpha^x\}$ in the subgame, which are realized with positive probabilities $p(\xi_\alpha^x, (\mu_i^x, \hat{\mu}_{-i}^x))$. Then we can rewrite the inequality (15):

$$\sum_\alpha p(\xi_\alpha^x, (\mu_i^x, \hat{\mu}_{-i}^x)) \cdot h_i^x(\xi_\alpha^x) > \sum_m p(\eta_m^x, \hat{\mu}^x) \cdot h_i^x(\eta_m^x). \tag{16}$$

Taking lemma 3 into account, the pair μ_i^x and $\hat{\mu}_i^D$ of player‘s i mixed strategies in Γ_x and Γ_D can be obtained as the result of decomposition of some strategy $\beta_i = (\hat{\mu}_i^D, \mu_i^x)$ in Γ. Moreovere:

$$H_i(\beta_i, \hat{\mu}_{-i}) = \sum_\alpha p(\bar{\eta}_x, \hat{\mu}^D) \cdot p(\xi_\alpha^x, (\mu_i^x, \hat{\mu}_{-i}^x)) \cdot h_i^x(\xi_\alpha^x) + \sum_k p(\chi_k, \hat{\mu}) \cdot h_i(\chi_k). \tag{17}$$

Now let us multiply both parts of inequality (16) onto positive value $p(\bar{\eta}_x, \hat{\mu}^D)$ and then add $\sum_k p(\chi_k, \hat{\mu}) \cdot h_i(\chi_k)$ to both parts of obtained vector inequality.

Taking (17) and (14) into account, we will finally have:

$$H_i(\beta_i, \hat{\mu}_{-i}) > H_i(\hat{\mu}_i, \hat{\mu}_{-i}).$$

This inequality contradicts (13).

Hence, the set $ME(\Gamma)$ in mixed strategies satisfies the time consistency property in n-person multicriteria extensive games with incomplete information. $\square$

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Waiting Time Costs in a Bilevel Location-Allocation Problem

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Abstract We present a two-stage optimization model to solve a location-allocation problem: finding the optimal location of new facilitites and the optimal partition of the consumers. The social planner minimizes the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity costs plus the distribution costs in the service regions. Theoretical and computational aspects of the location-allocation problem are discussed for the linear city and illustrated with examples.

Keywords: bilevel optimization, continuous facility location.

1. Introduction

Facility location problems deal with the question to locate some facilities in a continuous or discrete space by minimizing the total cost of opening sites and transporting goods or services to costumers (see, for example, Drezner, 1995, Love et al., 1988, Nickel and Puerto, 2005).

Several papers study single or multiple facility location, competitive location or dynamic location, and so on. In a Game Theory context, competitive models consider facilities competing for costumers and their objective is to maximize the market share they capture (allocation problem). The first competitive location model is in Hotelling, 1929 where the location of two duopolists whose decision variables are locations and prices is chosen. References for spatial competition can be found in Aumann's work (Aumann and Hart, 1992).

In a previous paper (Crippa et al., 2009), given the location of the facilitites, the authors considered the problem of splitting the costumers in such a way to minimize the waiting time effects and used optimal transportation tools. In another paper (Murat et al., 2009) the problem of finding the best partition of the costumers is considered together with the problem of finding the best location of the facilities and an algorithm procedure is provided. The authors minimize a total cost function in order to find at the same time the optimal costumer partition and the optimal facility location.

In this paper we present a bilevel approach to the problem: we look for the optimal location of the facilitites and also for the optimal partition of the costumers of the given market region. We extend the model studied in (Murat et al., 2009) by considering the waiting time inside the cost function in the spirit of the model studied in (Crippa et al., 2009).

More precisely, consider a distribution of citizens in an urban area in which a given number of services must be located. Citizens are partitioned in service regions such that each facility serves the costumer demand in one of the service regions. For a fixed location of all the services, every citizen chooses the service minimizing the total cost, i.e. the capacity acquisition cost plus the distribution cost (depending on the travel distance). In our model there is a fixed cost of each service depending on its location and an additional cost due to time spent in the queue of a service, depending on the amount of people waiting at the service, but also on the characteristics of the service (for example, its dimension). The objective is to find the optimal location of the services in the urban area and the related costumers partition. We present a two-stage optimization model to solve this location-allocation problem. The social planner minimizes the social costs, i.e. the fixed costs plus the waiting time costs, taking into account that the citizens are partitioned in the region according to minimizing the capacity costs plus the distribution costs in the service regions.

In Section 2 the linear and the planar models are presented; in Section 3 computational aspects and some examples are discussed; Section 4 contains concluding remarks.

2. The bilevel problem

Let Ω be a compact subset in $\mathcal{R}^2$. Each point $p = (x, y) \in \Omega$ has demand density $D(p)$ such that $\int_\Omega D(p)dp = 1$ with $dp = dxdy$. The problem is to locate n new facilities $p_1, ..., p_n$, $p_i = (x_i, y_i) \in \Omega$ for any $i \in N = \{1, 2, ..., n\}$. Facility p_i serves the consumers demand in the region $A_i \subseteq \Omega$: we have a partition of the set Ω, i.e. $\cup_{i=1}^n A_i = \Omega$ and $A_i \cap A_j \neq \emptyset$ for any $i \neq j$.

For any $i \in N$, we denote $\omega_i = \int_{A_i} D(p)dp$ the total demand within each service region A_i. Now we define for any $i \in N$:

1. $F_i(p_i)$ annualized fixed cost of facility i;
2. $a_i(p_i)$ annualized variable capacity acquisition cost per unit demand;
3. $C_i(p_i) = c\int_{A_i} d^2(p_i, p)D(p)dp$ is the distribution cost in service region A_i, being $d(\cdot,\cdot)$ the Euclidean distance in $\mathcal{R}^2$ and c the distribution cost per distance unit that we suppose to be constant in Ω;
4. $h_i(\omega_i)$ total cost, in term of time spent to be served, of consumers of region A_i using the service p_i.

We denote by $\mathcal{A}_n$ the set of all partitions in n sub-regions of the region Ω, $A = (A_1, ..., A_n) \in \mathcal{A}_n$ and $p = (p_1, ..., p_n) \in \Omega^n$.

Definition 1. Any tuple $< \Omega; p_1, ..., p_n; l, Z >$ is called a facility location situation, where $\Omega = [0, 1]$, $p_i \in \Omega$ for any $i \in N$; $l, Z : \Omega^n \times \mathcal{A}_n \to \mathcal{R}$ defined by

$$l(p, A) = \sum_{i=1}^{n} \left[F_i(p_i) + \omega_i h_i(\omega_i)\right], \tag{1}$$

$$Z(p, A) = \sum_{i=1}^{n} \left[\omega_i a_i(p_i) + c\int_{A_i} d^2(p_i, p)D(p)dp\right] \tag{2}$$

where ω_i is the total demand within service region A_i for any $i = 1, ..., n$, namely

$$\omega_i = \int_{A_i} D(p)dp. \tag{3}$$

Given a facility location situation, the goal is to find an optimal location for the facilities $p_1, ..., p_n$ and also an optimal partition $A_1, ..., A_n$ of the consumers in the market region Ω by minimizing the costs. We distinguish the total cost in a geographical part that is given by Equation 2 and in a social part that is given by Equation 1.

To this aim we propose a bilevel approach. Given the location of the new facilities, we search the optimal partition of the costumers. Then, we optimize another criterium to look for the optimal location of the facilties according to a bilevel formulation.

For a given location $p \in \Omega^n$ of the n facilities, the consumers have to decide which is the best facility to use: they minimize the costs given by the distributions costs, that depend on the distance from the chosen facility, plus the acquisition costs, that is the capacity acquisition cost of the facility supposed to be linear with respect to the density in the region where the chosen facility is. This is the geographical part given by Equation 2.

For any $p \in \Omega^n$, the optimal partition of the consumers in the set $\mathcal{A}_n$ will be a solution to the following lower level problem $LL(p)$:

$$\min_{A \in \mathcal{A}_n} Z(p, A). \tag{4}$$

Suppose that the problem $LL(p)$ has a unique solution for any $p \in \Omega^n$, let us call it $(A_1(p), ..., A_n(p)) = A(p)$. The function mapping to any $p \in \Omega^n$ the partition $A(p)$ represents for a given location of the new facilities, the best partition of the consumers that minimize their costs coming from the mutual distribution of the facilities and the costumers.

At this point the social planner proposes the best location of the n facilities in such a way that additional costs - that are social costs - as the fixed cost of each facility plus a cost due to the waiting time cost must be the lowest possible. These costs are given by Equation 1.

More precisely, the optimal location of the facilitites $\bar{p} \in \Omega^n$ solves the following upper level problem UL:

$$\min_{p \in \Omega^n} l(p, A(p)), \tag{5}$$

where for a given location p the optimal partition $A(p)$ of Ω is given by the unique solution of the problem $LL(p)$.

The problem UL is known as a bilevel problem, since it is a constrained optimization problem with the constraint that $A(p)$ is the solution of another optimization problem $LL(p)$ for any $p \in \Omega^n$.

Definition 2. Any $\bar{p}$ that solves the problem UL is an optimal solution to the bilevel problem.

In this case the *optimal pair* is $(\bar{p}, A(\bar{p}))$ where $\bar{p}$ solves the problem UL and $A(p)$ is the unique solution of the problem $LL(p)$ for each $p \in \Omega^n$.

Remark 1. In a Game Theory context, the solution of the upper level problem is called Stackelberg strategy and the pair solution of the bilevel problem as given in Definition 2 is called Stackelberg equilibrium (Başar and Olsder, 1995).

2.1. The linear city

We consider a linear region on the real line, i.e. a compact real interval Ω. Without loss of generality we normalize it and assume $\Omega = [0, 1]$. This assumption corresponds to concrete situations as the location of a gasoline station along a highway or the location of a railway station to improve the service to the inhabitants of the region.

Let $D(p)$ be the demand density s.t. $\int_0^1 D(p)dp = 1$ where $dp = dx$. We want to locate n facilities $p_i = x_i \in [0, 1]$ for any $i = 1, ..., n$ with $p_1 < p_2 < ... < p_n$. A partition $A = (A_1, ..., A_n)$ of the region $\Omega = [0, 1]$ is given by a real vector $\lambda = (\lambda_1, ..., \lambda_{n-1})$ such that $\lambda_i \in [p_i, p_{i+1}], i = 1, .., n - 1$. The partition in this case is: $A_1 = [0, \lambda_1[$,..., $A_n =]\lambda_{n-1}, 1]$. We denote $\lambda_0 = 0$ and $\lambda_n = 1$.

A linear facility location situation is a tuple $< \Omega; p_1, ..., p_n; l_1, Z_1 >$, where $\Omega = [0, 1]$, $p_i \in \Omega$ for any $i \in N$; $l_1, Z_1 : \Omega^n \times \mathcal{A}_n \to \mathcal{R}$ defined by

$$l_1(p, \lambda) = \sum_{i=0}^{n-1} [F_{i+1}(p_{i+1}) + \omega_{i+1} h_{i+1}(\omega_{i+1})] \tag{6}$$

$$Z_1(p, \lambda) = \sum_{i=0}^{n-1} \left[\omega_{i+1} a_{i+1}(p_{i+1}) + c \int_{\lambda_i}^{\lambda_{i+1}} d^2(p_{i+1}, p) D(p) dp \right] \tag{7}$$

where ω_i is the total demand within service region $A_i = [\lambda_{i-1}, \lambda_i]$ for any $i = 1, ..., n$, namely

$$\omega_i = \int_{\lambda_{i-1}}^{\lambda_i} D(p)dp. \tag{8}$$

Definition 3. Any $\bar{p}$ that solves the problem

$$\min_{p \in \Omega^n} l_1(p, \lambda(p)) \tag{9}$$

is an optimal solution to the bilevel problem, where for each $p \in \Omega^n$, $\lambda(p)$ is the unique solution of the problem $LL(p)$

$$\min_{\lambda \in [p_1, p_2] \times \ldots \times [p_{n-1}, p_n]} Z_1(p, \lambda) \tag{10}$$

In this case the *optimal pair* is $(\bar{p}, \lambda(\bar{p}))$ where $\bar{p}$ solves the problem UL and $\lambda(p)$ is the unique solution of the problem $LL(p)$ for each $p \in \Omega^n$.

We assume in the following that:

1. the demand density D is a continuous function on Ω s.t. $\int_0^1 D(p)dp = 1$;
2. h_i, F_i, a_i are continous functions on Ω for any $i = 1, ..., n$;
3. for any $p \in \Omega^n$, the problem $LL(p)$ has a unique solution $\lambda(p) \in \Omega^{n-1}$.

Proposition 1. *Under assumptions 1-3, the problem UL has at least a solution $\bar{p} \in \Omega^n$.*

Proof (of proposition). The function $Z_1(p, \lambda)$ is separable in λ since for any $i = 1, ..., n$

$$\omega_i = \int_{\lambda_{i-1}}^{p_i} D(p)dp + \int_{p_i}^{\lambda_i} D(p)dp \tag{11}$$

and by assumptions has a unique minimum point $\lambda_i(p) \in [p_i, p_{i+1}], i = 1, .., n-1$ for any $p \in \Omega^n$. The map $p \in \Omega^n \to \lambda(p) \in \Omega^{n-1}$ turns out to be a continuous function by using the Berge's theorem (Border, 1989).

The function $l_1(p, \lambda(p))$ is continuous and the problem UL admits at least a solution $\bar{p} \in \Omega^n$. $\square$

3. Numerical results

In this Section we present some computational results to solve the linear location-allocation problem. Our approach is based on Genetic Algorithms (GAs), a heuristic search technique modeled on the principle of evolution with natural selection. Namely, the main idea is the reproduction of the best elements with possible crossover and mutation. The detailed algorithm for a Stackelberg problem can be found in (D'Amato et al., 2012), and also in (D'Amato et al., 2011) in the case of non unique solution to the lower level problem.

The initial population is provided with a random seeding in the leader's strategy space. For each individual (or chromosome) of the leader population, a random population for the follower player is generated and a best reply search for the follower player is made. The follower player best reply passes to the leader: the leader population is sorted under objective function criterium and a mating pool is generated. Now a second step begins and a common crossover and mutation operation on the leader population is performed. Again the follower's best reply should be computed, in the same way described above. This is the kernel procedure of the genetic algorithm that is repeated until a terminal period is reached or an exit criterion is met.

For the algorithm validation we consider the parameters as specified in Table 1.

Table1: GA details

Parameter	Value
Population size (-)	50
Crossover fraction (-)	0.90
Mutation fraction (-)	0.10
Parent sorting	Tournament between couple
Mating Pool (%)	50
Elitism	no
Crossover mode	Simulated Binary Crossover (SBX)
Mutation mode	Polynomial

3.1. Test cases

Example 1. (*Uniform density*) We want to locate two new facilities in the linear market region $[0,1] \subset \mathcal{R}$ where the consumers are uniformly distributed ($D(p) = 1$ for any $p \in [0,1]$). The generic partition is $A_1 = [0, \lambda[,\ A_2 =]\lambda, 1]$ for $\lambda \in [0,1]$. Then the density of each part is $\omega_1 = \lambda$ and $\omega_2 = (1-\lambda)$. In this example the fixed costs, the acquisition costs, the distribution costs and the waiting time costs are respectively for $\varepsilon > 0$:

$$F_1(p_1) = p_1^2,\ F_2(p_2) = p_2/4, \tag{12}$$

$$a_1(p_1) = p_1^2,\ a_2(p_2) = p_2^2, \tag{13}$$

$$C_1(p_1) = 3\int_0^\lambda (p_1 - p)^2 dp,\ C_2(p_2) = 3\int_\lambda^1 (p_2 - p)^2 dp, \tag{14}$$

$$h_1(t) = (1+\varepsilon)t,\ h_2(t) = t. \tag{15}$$

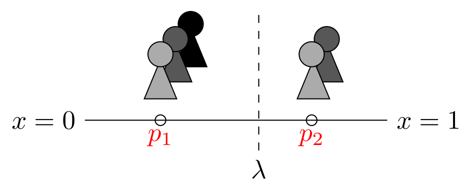

Figure1: Location of two facilities in the linear city.

Let us consider the facility location situation $< [0,1]; p_1, p_2; l_1, Z_1 >$ where

$$l_1(p_1, p_2, \lambda) = p_1^2 + p_2/4 + (1+\varepsilon)\lambda^2 + (1-\lambda)^2, \tag{16}$$

$$Z_1(p_1, p_2, \lambda) = p_1^2\lambda + p_2^2(1-\lambda) + (\lambda - p_1)^3 + p_1^3 + (1-p_2)^3 - (\lambda - p_2)^3. \tag{17}$$

Our problem is to find $p_1, p_2 \in [0,1]$ with $0 \leq p_1 < p_2 \leq 1$ that solves

$$\min_{\lambda \in [p_1, p_2]} Z_1(p_1, p_2, \lambda). \tag{18}$$

The unique solution is

$$\lambda(p_1, p_2) = \begin{cases} \frac{2(p_1+p_2)}{3} & \text{if } 2p_1 \leq p_2, \\ p_2 & \text{if } 2p_1 > p_2. \end{cases} \tag{19}$$

The social planner problem is

$$\min_{p_1, p_2 \in [p_1, p_2]} l(p_1, p_2, \lambda(p_1, p_2)). \tag{20}$$

It is possible to compute that for $\varepsilon < \frac{5}{4}$ the solution is

$$(\bar{p}_1, \bar{p}_2) = \left(\frac{1}{8}, \frac{31 - 4\varepsilon}{32(2+\varepsilon)}\right), \tag{21}$$

and then

$$\bar{\lambda} = \frac{13}{16(2+\varepsilon)}. \tag{22}$$

For $\varepsilon = 1$ the analytical solution is:

$$(\bar{p}_1, \bar{p}_2) = (\frac{1}{8}, \frac{27}{96}) = (0.125, 0.2812), \quad \bar{\lambda} = \frac{13}{48} = 0.2708.$$

Remark 2. In the perfect symmetric situation where $F_1 = F_2 = 0$ and $\varepsilon = 0$, the facility location situation is $< [0,1]; p_1, p_2; l_1, Z_1 >$ where

$$l_1(p_1, p_2, \lambda) = \lambda^2 + (1-\lambda)^2, \tag{23}$$

$$Z_1(p_1, p_2, \lambda) = p_1^2\lambda + p_2^2(1-\lambda) + (\lambda - p_1)^3 + p_1^3 + (1-p_2)^3 - (\lambda - p_2)^3. \tag{24}$$

In this case $\bar{\lambda} = \frac{1}{2}$ gives the optimal partition. Optimal location is any pair in the set

$$\{(p_1, 3/4 - p_1), p_1 \in [0, \frac{1}{4}]\} \cup \{(p_1, 1/2), p_1 \in]1/4, 1/2[\}. \tag{25}$$

Test cases.
Uniform density. In the case of uniform density, i.e. $D(x) = 1$ for any $x \in [0,1]$, with $\varepsilon = 1$, the numerical computation gives:

$$(\bar{p}_1, \bar{p}_2) = (0.1238, 0.2811), \quad \bar{\lambda} = 0.2694.$$

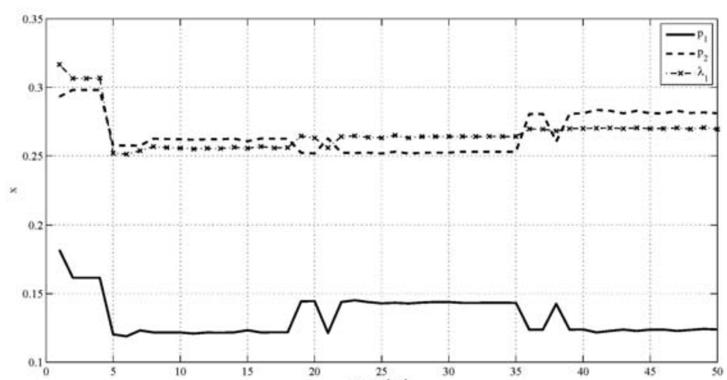

Figure2: History of implementation in the linear city with uniform density.

The convergence histories of the linear city with uniform density are reported in Figure 2.

Beta-shaped density. In the case of beta-shaped density as in Figure 3, i.e.

$$D(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du},$$

for any $x \in [0,1]$, $\alpha = 4,\ \beta = 4$, with $\varepsilon = 1$, we have the following results:

$$(\bar{p}_1, \bar{p}_2) = (0.1200, 0.4009), \quad \bar{\lambda} = 0.3472.$$

The convergence histories of the linear city with beta-shaped density are reported in Figure 4.

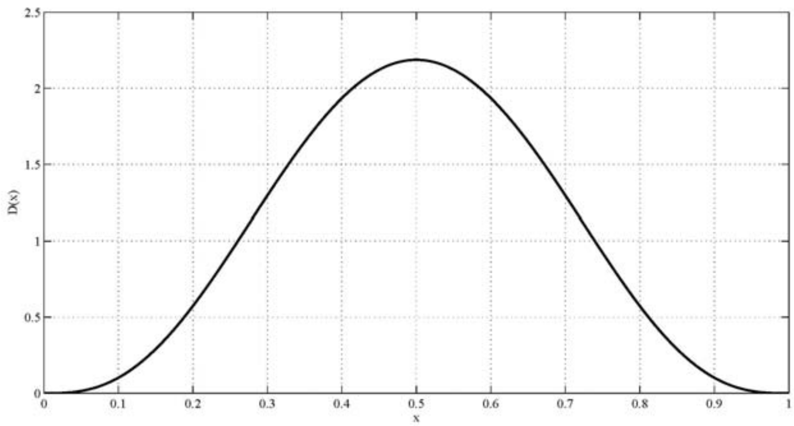

Figure3: A beta-shaped density function.

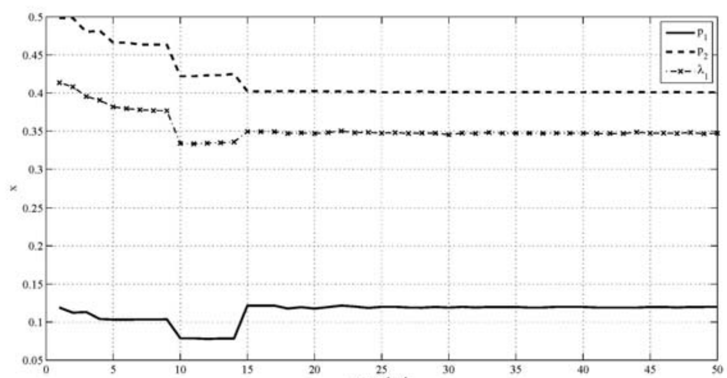

Figure4: History of implementation in the linear city with beta-shaped density.

Two beta-shaped density. In the case of two beta distributions summed on a partly shared interval as in Figure 5,

$$D(x) = \frac{x_1^{\alpha-1}(1-x_1)^{\beta-1} + x_2^{\alpha-1}(1-x_2)^{\beta-1}}{\int_0^k u_1^{\alpha-1}(1-u_1)^{\beta-1}du_1 + \int_{1-k}^1 u_2^{\alpha-1}(1-u_2)^{\beta-1}du_2}$$

where $x_1 \in [0,k]$ and $x_2 \in [1-k,1]$, with $k = 0.65$, $\alpha = 4$, $\beta = 4$, with $\varepsilon = 1$, we have the following results:

$$(\bar{p}_1, \bar{p}_2) = (0.1251, 0.3509), \quad \bar{\lambda} = 0.3176$$

The convergence histories of the linear city with two beta-shaped density are reported in Figure 6.

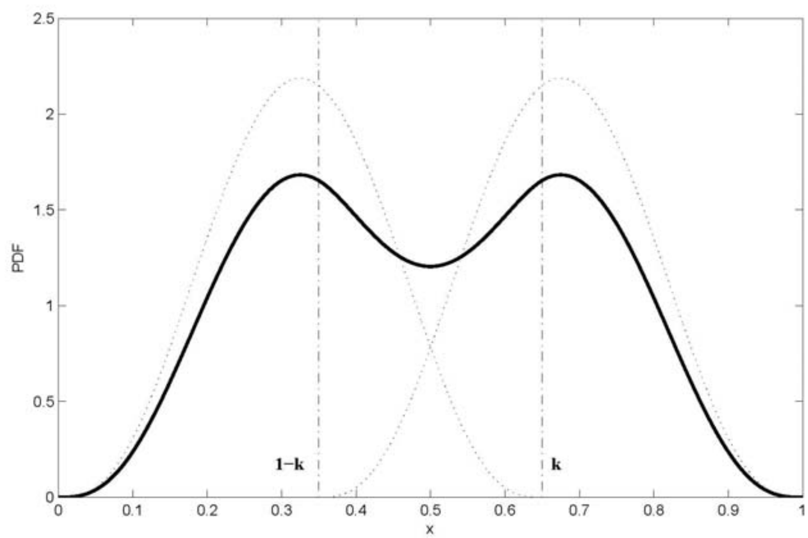

Figure5: Two beta distribution summed density functions.

A summary of the analyzed test cases is reported in Table 2.

Table2: Test cases

Distribution	$\bar{p}_1$	$\bar{p}_2$	$\bar{\lambda}$
Uniform	0.1238	0.2811	0.2694
Beta	0.1200	0.4009	0.3472
Two-beta	0.1251	0.3509	0.3176

4. Concluding Remark

The problem studied in this paper has a lot of computational difficulties. An algorithm based on sections of the elements $A_1, ..., A_n$ of the partitions is given in

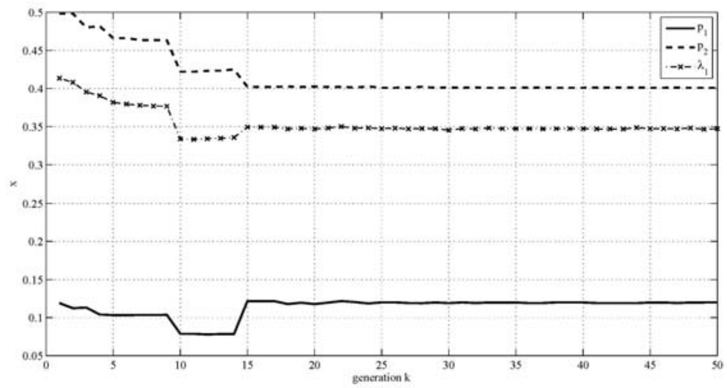

Figure6: History of implementation in the linear city with two beta-shaped density.

(Murat et al., 2009) for a similar problem formulated as an optimization problem not by considering several hierarchical levels and without the waiting time costs. The algorithm in (Murat et al., 2009) uses Voronoi diagrams. In this paper we appoached the linear facility problem by using a genetic algorithm. The location in a planar region together with computational aspects will be studied in a future research. Also the circular region case (see, for example, Mazalov and Sakaguchi, 2003) would be interesting to investigate.

For a given facility location situation $< \Omega; p_1, ..., p_n; l, Z >$, it may happen also that the lower level problem $LL(p)$ has more that one solution. Let us call $\mathcal{A}(p)$ the set of the solutions to $LL(p)$ for any p. In this case we can define the upper level problem in a different way. In a pessimistic framework, the social planner could use the so called *security strategy* in order to prevent the worst that can happen when the consumers organize themselveves in any of the partitions indicated in the set $\mathcal{A}(p)$.

More precisely, the optimal location of the facilitites $\bar{p} \in \Omega^n$ solves the following upper level problem UL^s:

$$\min_{p \in \Omega^n} \max_{A \in \mathcal{A}(p)} l(p, A). \tag{26}$$

Definition 4. Any $\bar{p}$ that solves the problem UL^s is called a security strategy to the problem UL^s.

The existence and properties of the security strategies will be investigated in the future.

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Separating and Pooling Incentive Mechanisms of Ecological Regulation: The Cases of Developed and Developing Countries ⋆

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Abstract A model of contract theory is studied, where the objective functions of a regulation body and firms of two types involve ecological variables. It is shown that the way of working of the regulation mechanism (unifying or pooling) depends on both political conditions (regulators of what type set mechanism and contracts), and on economical conditions (distinction between "dirty" and "green" firms in efficiency and a degree of their spreading in the economy). Under small difference in a parameter values characterizing the types of firms it appears that if (what seems to be typical for many developing and transition economies) a use of "dirty" technologies raises the rentability of firms and the part of "dirty" firms in economy is great then the pooling (i.e., in some sense, non-market) contract mechanism is chosen more often. Under conditions which seem to be typical for developed countries (relatively more efficient "green" firms), a choice of separating (in a more degree market) mechanism can be expected.

Keywords: Menu of contracts, pooling contract, ecological regulation, developed and developing countries.

AMS Classification: 91A, 91B, 94A, 93C, 49B.
JEL Classification: Q57, C72, C73, K22, P48.

1. Preface

A part of the global problem of stabilization of environment is connected with ensuring of effective ecological regulation in transition and developing economies, where a considerable part of the world industrial production is concentrated. In 2004 the share of seven main "emerging" economies (E7: China, India, Brazil, Russia, Mexico, Indonesia, Turkey) in global carbonic dioxide emission was 32.1%, and according to forecasts it will increase up to 42.6% by 2025 year and up to 49% by 2050 (Hawksworth, 2006). According to (Davis and Caldeira, 2010), the main commodities exporters in the world whose production is related with the atmosphere pollution in the present time are China, Russia, Middle East countries, South Africa countries, Ukraine, India, Malaise, Thailand, Thai-vane, Venezuela.

Researchers usually explain modest results of economic policy in Russia and other transitional economies, and, in particular, of ecological policy, by "inherited"

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manners of behavior and institutions, as well as by conflicts between new formal and old non-formal institutions. However, there is another possibility: "new" economies possess purely economic peculiarities that lead to serious differences in work of those institutional mechanisms which showed themselves quite good in developed countries.

If in industrial countries the same firms that inflict the least damage to environment are in the same time the most effective in the sense of rentability, then in many developing and transition economies, on the contrary, many firms can obtain a large economic gain by inflicting a direct or indirect pollution.

Laffont (Laffont, 2000) investigated a model of ecological regulation that rather exactly corresponds to economic situation in industrial countries. In this model firms-monopolists are considered which possess cost function as follows:

$$C(\theta, d) = \theta(K - d),$$

where $K > 0$ is some common for all the firms constant, $\theta > 0$ is a characteristic of costs that is a private information of a firm (the type of firm), $d > 0$ is a level of pollution allowed for firms of that type (chosen by the firm from a menu of contracts proposed by the regulator or definitely established by the regulator). It follows from the formula that if there are two types of firms, $\underline{\theta} < \bar{\theta}$, under a possibility to increase the level of pollution d the firm of type $\underline{\theta}$ (it may be interpreted as "green") receives a smaller cost decrease than the firm of type $\bar{\theta}$ ("dirty").

The regulating body possessing an information about costs of the types and about their share (frequency) in the economy but possessing no information about a type of a concrete firm assigns either a pooling contract or a menu of contracts $M = \{(\underline{t}, \underline{d}), (\bar{t}, \bar{d})\}$ (where $\underline{t}, \bar{t}$ are the sizes of transfers, $\underline{d}, \bar{d}$ are allowed pollution levels) from which a firm chooses an optimal for itself contract. In the Laffont model the firm of type $\underline{\theta}$ is economic efficient and receives an information rent; the origin of the latter is related to a possibility for the firm to "pretend" to belong to other type.

Three types of regulators were considered, they differ by objective functions: a social maximizer, an interested majority and an disinterested majority; the interested majority is found to be the most effective regulator from the point of view of decreasing the pollution levels .

Matveenko (Matveenko, 2010) has proposed a more general model with a cost function:

$$C(\theta, d) = \kappa(\theta) - \theta d, \tag{1}$$

where $\kappa(\theta) > 0$. If there are two type of firms, $\underline{\theta} < \bar{\theta}$ then it is natural to consider as *an index of relative economical efficiency* the value

$$\tilde{K} = \frac{\kappa(\bar{\theta}) - \kappa(\underline{\theta})}{\Delta\theta},$$

where $\Delta\theta = \bar{\theta} - \underline{\theta}$. The relative efficiency of a "dirty" firm may increase both by increasing of differential $\Delta\theta$ and by decreasing the value $\kappa(\bar{\theta})$, that may be interpreted as investments in quality of product (e.g., the costs of R&D and modernization). Negative values of $\widetilde{K}$ are permitted. For "small" values of $\tilde{K}$ the firm of type $\bar{\theta}$ ("dirty") proves to be a a rent receiver, and for "high" values of $\tilde{K}$ the type $\underline{\theta}$ ("green") does. For "intermediate" values of $\tilde{K}$, no type of firms can capture a rent.

The notions of "small" and "high" $\tilde{K}$ are defined more precisely in dependence on which type regulator is in power and forms the menu of contracts.

In a case typical for developing and transitional economies, when the share (frequency) of firms able to receive an advantage from pollution is relatively large and these firms are relatively effective ($\tilde{K}$ is "small"), the interested sides being in power allow an extremely high pollution level for firms of type $\underline{\theta}$; moreover, not a separating mechanism with a free choice from a menu of contracts is used but a pooling mechanism i.e. an assignment of a unique common contract. That implies (outside the frame of the model) a more high degree of the state intervention into economy and more narrow relations between the regulator and the firms which may lead to a higher degree of corruption. All this takes place under the same "standard" institutions which are relatively successful in solving the ecological regulation problem in developed countries where economic condition are different ($\tilde{K}$ is "high").

In this paper the study of the model (Matveenko, 2010) is continued and the main attention is paid to the question: what type of the mechanism (pooling or separating) will be chosen under different political and economical conditions? The research is done under an assumption of small $\Delta\theta$. Two situations are under consideration:

(a) the type of mechanism is defined by the society whilst the decision, in frame of this mechanism, is made by a regulator (interested or disinterested sides),

(b) both the type of the mechanism and the decision about pollution levels are defined by a regulator.

We show that under conditions which seem to be typical for developing and transition economies ("dirty" firms are relatively effective and their share in the economy is relatively high), one ought in more degree expect a pooling (i.e. non-market) mechanism assignment.

In section 2 a description of the model is given. In section 3 the equilibrium pollution levels in different cases are found. In section 4 a comparison of the separating and the pooling mechanisms is done. Section 5 concludes.

2. The basic model

Let a fulfillment of a project having social value S be realized by a firm which carries pure costs (1), where $\kappa(.) > 0$, d is a pollution level allowed to the firm, θ is a characteristic of costs which is a private information of the firm (the type of the firm), and θ takes two values: $\underline{\theta}$ with probability ν and $\bar{\theta}$ with probability $(1-\nu)$, and $\underline{\theta} < \bar{\theta}$.

Denote through t a pure transfer received by the firm. For $t > 0$ it is actually a transfer, and for $t < 0$ the magnitude $(-t)$ represents a tax paid by the firm. The rent received by the firm is

$$U = t - C(\theta, d).$$

We admit a possibility of $C(\theta, d) < 0$, i.e. of receiving a pure profit by the firm (it may be supposed, in sake of simplicity, that the pure profit arises at the expense of an export activity, i.e. it does not lie down on the shoulders of the consumers). The firm will execute the project if $U \geq 0$. In contract theory this condition is known as *individual rationality, IR.*

A social evaluation of a pollution harm is $V(d)$ where $V'(.) > 0$, $V''(.) > 0$. The welfare of the consumers is equal to

$$S - V(d) - (1+\lambda)t.$$

In (Laffont, 2000) the parameter λ is interpreted as social costs per unit of transfer. We, admitting also a possibility of a tax, tract $1+\lambda$ broader as a rate of returns which characterizes the advantage of using in another projects the means which the society loses in form of transfer or gains in form of tax from firm. We assume that $\lambda > 0$ is constant; the passage to an assumption that λ is a random value doesn't change the character of results.

The social welfare consist of the consumers' welfare and the rent:

$$S - V(d) - (1+\lambda)t + U = S - V(d) - (1+\lambda)(\kappa(\theta) - \theta d) - \lambda U.$$

Under a perfect information the social welfare maximization results in zero rent, and for firms of types $\underline{\theta}$ and $\bar{\theta}$, correspondingly, the pollution levels $\underline{d}^*$, $\bar{d}^*$ are assigned such that

$$V'(\underline{d}^*) = (1+\lambda)\underline{\theta}, \tag{2}$$
$$V'(\bar{d}^*) = (1+\lambda)\bar{\theta}.$$

Under an imperfect information, when the type of firms is unknown to the regulator, if the *separating regulation mechanism* acts the regulator proposes to a firm a menu of contracts

$$M = (\underline{t}, \underline{d}), (\bar{t}, \bar{d}),$$

satisfying (1) the conditions of *incentives compatibility, IC*, the sense of which is that no firm can receive a gain by "pretending" to be a firm of another type:

$$\underline{t} - C(\underline{\theta}, \underline{d}) \geq \bar{t} - C(\underline{\theta}, \bar{d}), \tag{3}$$

$$\bar{t} - C(\bar{\theta}, \bar{d}) \geq \underline{t} - C(\bar{\theta}, \underline{d}), \tag{4}$$

and (2) conditions of IR which have been already mentioned:

$$\underline{t} - C(\underline{\theta}, \underline{d}) \geq 0, \tag{5}$$

$$\bar{t} - C(\bar{\theta}, \bar{d}) \geq 0. \tag{6}$$

Besides, the menu of contracts M maximizes the regulator's objective function, in which transfers enter with a minus sign, i.e. the regulator is interested in cutting down transfers. In (Matveenko, 2010) it is is proven that the optimal menu of contracts satisfying the conditions IC and IR possesses the following properties:

1) a necessary and sufficient condition of receiving a rent by a firm of type $\underline{\theta}$ is the inequality $\tilde{K} > \bar{d}$ (the case of "*large*" $\tilde{K}$);

2) a necessary and sufficient condition of receiving a rent by a firm of type $\bar{\theta}$ is the inequality $\tilde{K} < \underline{d}$ (the case of "*small*" $\tilde{K}$);

3) if $\underline{d} \leq \tilde{K} \leq \bar{d}$ (the case of "*intermediate*" $\tilde{K}$) then no type of firms may obtain rent.

In the case of "large" $\tilde{K}$ a firm of type $\bar{\theta}$ receives no rent, and the rent received by the firm of type $\underline{\theta}$ is equal to

$$\underline{U} = \bar{t} - C(\underline{\theta}, \bar{d}) = \Delta\theta(\tilde{K} - \bar{d}).$$

In the case of "small" $\tilde{K}$ a firm of type $\underline{\theta}$ receives no rent, and a firm of type $\bar{\theta}$ receives the rent

$$\bar{U} = \underline{t} - C(\bar{\theta}, \underline{d}) = \Delta\theta(\underline{d} - \tilde{K}).$$

Thus, the rent depends on the pollution level of another (receiving no rent) type of firm, but the dependence under a "large" $\tilde{K}$ is negative and under a "small" $\tilde{K}$ is positive. This fact, essentially, defines the difference in pollution levels which the interested sides being in power set under different economic conditions.

Let us assume that the interested sides are in power with probability p, and the disinterested sides are in power with probability q, and that each of the sides being in power receives a share $\alpha^* > 1/2$ of the consumers welfare. An analogous assumption in (Laffont, 2000) is motivated by supposing that, under conditions of democracy, a majority of population comes to power and the majority is always α^*. Referring the types of the regulator we use in the paper terms (Laffont, 2000): *disinterested majority*, or *majority-1* when we speak about the disinterested sides in power, and *interested majority*, or *majority-2* when we speak about the interested sides in power. For us these are only the terms to distinguish regulators' types.

3. Decisions of the regulator

In this section we indicate equilibrium pollution levels being included into the menu of contracts (in case, when the regulator uses a separating mechanism) or being set (if the regulator uses a pooling mechanism). The knowledge of these pollution levels will be needed in Section 4 for a comparison of separating and unifying mechanisms.

3.1. Separating mechanism

Disinterested majority makes decision Let the firm of type $\overline{\theta}$ receive a rent (the case of "small" $\tilde{K}$). The objective function of the majority-1 takes the form

$$\alpha^* E[S - V(d) - (1+\lambda)t] = \alpha^*[\nu(S - V(\underline{d}) - (1+\lambda)(\kappa(\underline{\theta}) - \underline{\theta d})) + \\ +(1-\nu)(S - V(\bar{d}) - (1+\lambda)(-\bar{\theta}\bar{d} + \kappa(\underline{\theta}) + \Delta\theta\underline{d}))]. \tag{7}$$

Maximizing this function the majority-1 includes to the menu of contracts the pollution level $\overline{d}^*$ and the level $\underline{d}_1$ satisfying the equation

$$V'(\underline{d}_1) = (1+\lambda)\underline{\theta} - (1+\lambda)\frac{1-\nu}{\nu}\Delta\theta. \tag{8}$$

This menu of contracts is feasible only under $\tilde{K} < \bar{d}_1$. The latter inequality is the condition defining in that case the notion of "small" $\tilde{K}$.

Let firm of type $\underline{\theta}$ receive a rent (the case of "large" $\tilde{K}$). Analogously, the majority-1 inserts into the menu pollution levels $\underline{d}^*$ and $\bar{d}_1$, where

$$V'(\bar{d}_1) = (1+\lambda)\bar{\theta} + (1+\lambda)\frac{\nu}{1-\nu}\Delta\theta. \tag{9}$$

For the feasibilty of the menu of contracts the inequality $\tilde{K} > \bar{d}_1$ has to be fulfilled (this is an identifer of the "large" $\widetilde{K}$).

In case, when no type of firm receives a rent (the case of "intermediate" $\tilde{K}$), under $\underline{d}^* \leq \tilde{K} \leq \bar{d}^*$ the menu of contracts with pollution levels $\underline{d}^*$, $\bar{d}^*$ is optimal. Under $\underline{d}^* \leq \tilde{K} \leq \bar{d}_1$ the menu of contracts includes pollution levels $\underline{d} = \underline{d}^*$, $\bar{d} = \tilde{K}$. Under $\underline{d}_1 \leq \tilde{K} \leq \underline{d}^*$ the pollution levels $\underline{d} = \tilde{K}$, $\bar{d} = \bar{d}^*$ are used.

Interested majority makes decision If the firm of type $\overline{\theta}$ receives a rent (the case of "small" $\widetilde{K}$) then the objective function of the majority-2 has the form:

$$\alpha^*\nu[S - V(\underline{d}) - (1+\lambda)(\kappa(\underline{\theta} - \underline{\theta d}))+$$
$$+(1-\nu)(S - V(\bar{d}) - (1+\lambda)(\kappa(\overline{\theta}) - \bar{\theta}\bar{d}) - (1+\lambda - 1/\alpha^*)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \Delta\theta\underline{d}))]. \quad (10)$$

Maximization results in the pollution level $\overline{d}^*$ and the level $\underline{d}_2$ satisfying the following equation

$$V'(\underline{d}_2) = (1+\lambda)\underline{\theta} - \left(1 + \lambda - \frac{1}{\alpha^*}\right)\frac{1-\nu}{\nu}\Delta\theta. \quad (11)$$

This menu of contracts is feasible only if $\widetilde{K} < \underline{d_2}$. One more feasibility condition is the restriction on model parameters:

$$1 + \lambda > \frac{1-\nu}{\alpha^*}$$

(this inequality is equivalent to $\underline{d}_2 < \overline{d}^*$).

If the firm of type $\underline{\theta}$ receives the rent (the case of "large" $\widetilde{K}$) then, analogously, the majority-2 chooses the pollution level $\underline{d}^*$ and the level $\overline{d}_2$ such that

$$V'(\overline{d}_2) = (1+\lambda)\underline{\theta} - \left(1 + \lambda - \frac{1}{\alpha^*}\right)\frac{\nu}{1-\nu}\Delta\theta. \quad (12)$$

For feasibility the inequality $\widetilde{K} > \overline{d_2}$ is required. Besides, the following condition on the parameters has to be filfilled:

$$1 + \lambda > \frac{\nu}{\alpha^*}$$

(this is equivalent to $\underline{d}^* < \overline{d}_2$).

If no type of firms receives rent (the case of "intermediate" $\widetilde{K}$) and

$$1 + \lambda > \frac{1}{\alpha^*},$$

then for $\underline{d}^* \leq \widetilde{K} \leq \overline{d}^*$ the menu of contracts with pollution levels $\underline{d}^*$, $\overline{d}^*$ is optimal; under $\overline{d}^* < \widetilde{K} \leq \overline{d}_2$ the menu of contracts includes pollution levels $\underline{d} = \underline{d}^*$, $\overline{d} = \widetilde{K}$; and under $\underline{d}_2 \leq \widetilde{K} < \underline{d}^*$ the pollution levels $\underline{d} = \widetilde{K}$, $\overline{d} = \overline{d}^*$ will be used.

If the value $1 + \lambda - 1/\alpha^*$ is negative but is not too large by its absolute value, so that

$$\underline{d}^* < \underline{d}_2 < \overline{d}_2 < \overline{d}^*,$$

then under $\underline{d}_2 \leq \widetilde{K} \leq \overline{d}_2$ the regulator includes into the menu of contracts pollution levels $\underline{d}^*$ $\overline{d}^*$.

3.2. Unifying mechanism

Under definite conditions (see Section 4) it is advantageous for regulator to use a pooling mechanism instead of a separating menu of contracts. This may serve an explanation of comparatively low spreading of market regulation mechanisms in developing and transitional economies in comparison with developed countries.

Under a *pooling regulation mechanism* the regulator proposes only one (common for all firms) contract (t, d). Conditions IC now have no sense, but there ought to hold the conditions IR and, thus,

$$t = \max\{C(\underline{\theta}, d), C(\overline{\theta}, d)\}.$$

The rent $U = t - C(\theta, d)$ will be received by that type of firm for which costs are less. Under $\widetilde{K} < d$ the firm of type $\overline{\theta}$ receives a rent, and under $\widetilde{K} > d$ the firm of type $\underline{\theta}$ does, and, besides, in both cases the rent is equal to $|\widetilde{K} - d|\Delta\theta$. The rent is absent in a single case when $\widetilde{K} = d$.

Disinterested majority makes decision In case of "small" $\widetilde{K}$ when a rent is received by the firm of type $\overline{\theta}$ the majority-1 maximizes the function

$$\alpha^*[S - V(d) - (1+\lambda)E(\kappa(\theta) - \theta d) - (1-\nu)(1+\lambda)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \Delta\theta d)], \quad (13)$$

and solution is $d_1^s = \underline{d}^*$. A "smallness" $\widetilde{K}$ is realized as $\widetilde{K} < \underline{d}^*$.

In case of "large" $\widetilde{K}$ when the firm of type $\underline{\theta}$ receives a rent the majority-1 maximizes the function

$$\alpha^*[S - V(d) - (1+\lambda)E(\kappa(\theta) - \theta d) - \nu(1+\lambda)(\kappa(\overline{\theta}) - \kappa(\underline{\theta}) + \Delta\theta d)],$$

and solution is $d_1^h = \overline{d}^*$. "Large" $\widetilde{K}$ means $\widetilde{K} > \overline{d}^*$.

Disinterested majority makes decision In case of "small" $\widetilde{K}$ when the firm of type $\overline{\theta}$ receives a rent the majority-2 maximizes the function

$$\alpha^*[S - V(d) - (1+\lambda)E(\kappa(\theta) - \theta d) - (1-\nu)\left(1 + \lambda - \frac{1}{\alpha^*}\right)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \Delta\theta d)] \quad (14)$$

and sets such pollution level d_2^s that

$$V'(d_2^s) = (1+\lambda)[\nu\underline{\theta} + (1-\nu)\overline{\theta}] - (1-\nu)\left(1 + \lambda - \frac{1}{\alpha^*}\right)\triangle\theta] = (1+\lambda)\underline{\theta} + \frac{1}{\alpha^*}(1-\nu)\triangle\theta. \quad (15)$$

"Small" $\widetilde{K}$ means $\widetilde{K} < d_2^s$.

In case of "large" $\widetilde{K}$ when the firm of type $\underline{\theta}$ receives the rent the majority-2 maximizes the function

$$\alpha^*[S - V(d) - (1+\lambda)E(\kappa(\theta) - \theta d) - \nu\left(1 + \lambda - \frac{1}{\alpha^*}\right)(\kappa(\overline{\theta}) - \kappa(\underline{\theta}) - \Delta\theta d)],$$

and sets the pollution level d_2^h such that

$$V'(d_2^h) = (1+\lambda)[\nu\underline{\theta} + (1-\nu)\overline{\theta}] + \nu\left(1 + \lambda - \frac{1}{\alpha^*}\right)\triangle\theta] = (1+\lambda)\overline{\theta} - \frac{1}{\alpha^*}\nu\triangle\theta.$$

"Large" $\widetilde{K}$ means $\widetilde{K} > d_2^h$.

In the case of "small" $\widetilde{K}$ the least pollution level is set by the majority-1, and the greatest by the majority-2. This comparison is correct under $\widetilde{K} < \underline{d}^*$. In this case the pollution levels are such that

$$\underline{d}^* = d_1^s < d_2^s.$$

In contrary, in the case of "large" $\widetilde{K}$ the least pollution level is set by the majority-2, and the greatest by the majority-1. This comparison is correct under $\widetilde{K} > \overline{d}^*$. In this case pollution levels are such that

$$d_2^h < d_1^h = \overline{d}^*.$$

As far as the derivative $V'(.)$ must be positive we must put some additional restrictions on model parameters:

$$(1+\lambda)\overline{\theta} > \frac{1}{\alpha^*}\nu \bigtriangleup \theta,$$
$$\underline{\theta} > \frac{1-\nu}{\nu} \bigtriangleup \theta,$$
$$(1+\lambda)\overline{\theta} > (\frac{1}{\alpha^*} - 1 - \lambda)\frac{\nu}{1-\nu} \bigtriangleup \theta.$$

what, as is readily seen, equivalent to the following conditions:

$$\nu\overline{\theta} > \bigtriangleup \theta,$$
$$(1+\lambda)\overline{\theta} > \frac{1}{\alpha^*}\nu \bigtriangleup \theta + (1+\lambda)\nu\underline{\theta}.$$

Decision of majority-2 in the case typical for developing and transition economies For us the most interesting is the case

$$\frac{\nu}{\alpha^*} < 1 + \lambda < \frac{1-\nu}{\alpha^*}, \tag{16}$$

which seems to be typical for developing and transition countries, where the share $1-\nu$ of firms of type $\overline{\theta}$ in the economy is large. Note, that the left hand side of the inequality (16) means the admissibility of the separating mechanism under $\widetilde{K} > \overline{d}_2$ (see Section 3.1.2). The right hand side of (16) means a violation of conditions of feasibility of the pooling mechanism under $\widetilde{K} < \underline{d}_2$. Possible pollution levels are linked by the relation:

$$\underline{d}^* < \overline{d}_2 < \overline{d}^* < d_2^s < \underline{d}_2.$$

Hence, in a rather narrow interval $\overline{d}_2 < \widetilde{K} < d_2^s$ the feasibility conditions allow the majority-2 to employ both the pooling and the separating mechanisms. This case is especially interesting because the separating mechanism is employed through "large" $\widetilde{K}$ while the pooling mechanism is employed through "small" $\widetilde{K}$. In other words, the same value $\widetilde{K}$ appears to be "large" for the separating mechanism and "small" for the pooling mechanism. The mechanism for which the objective function is greater will be chosen; this depends, in particularly, on the kind of function $V(.)$.

Consider the case of quadratic function $V(d) = d^2$. We are interested, whether the type of the mechanism chosen by the majority-2 corresponds to the society's interests.

Theorem 1. *Let ν be small enough. If $\widetilde{K}$ is close to d_2^s and if $\widetilde{K} < d_2^s$ then the majority-2 will choose the separating mechanism and include into the menu of contracts the pollution levels $\underline{d}^*$ and d_2^s. This choice corresponds to the interests of the whole society. If $\widetilde{K}$ is close to $\overline{d}_2$ and $\widetilde{K} > \overline{d}_2$ then the majority-2 will choose the pooling mechanism and set the pollution level $\overline{d}_2$. This choice doesn't correspond to* *the interests of the society as the whole. Besides, the pollution levels are bound by the inequalities:*

$$\underline{d}^* < \overline{d}_2 < d_2^s.$$

Proof. The objective function of the majority-2 under the separating mechanism in the case of "large" $\widetilde{K}$ has the form:

$$W_2^{sep} = \alpha^*[\nu(S - V(\underline{d}) - (1+\lambda)(\kappa(\underline{\theta} - \underline{d\theta}) - \left(1 + \lambda - \frac{1}{\alpha^*}\right)(\kappa(\overline{\theta}) - \kappa(\underline{\theta}) - \overline{d}\Delta\theta)) +$$

$$+(1-\nu)(S - V(\overline{d}) - (1+\lambda)(\kappa(\overline{\theta}) - \overline{d\theta}))],$$

and under the unifying mechanism in the case of "small" $\widetilde{K}$ it has the form:

$$W_2^{un} = \alpha^*[\nu(S - V(d) - (1+\lambda)(\kappa(\underline{\theta}) - d\underline{\theta})) +$$

$$+(1-\nu)(S - V(d) - (1+\lambda)(\kappa(\overline{\theta}) - d\overline{\theta}) - \left(1 + \lambda - \frac{1}{\alpha^*}\right)(d\Delta\theta - \kappa(\overline{\theta} + \kappa(\underline{\theta}))))].$$

We will consider the limit behavior of the objective functions under $\nu \to 0$. Strict inequalities for the limit values will be held for sufficiently small values ν. We obtain

$$L^{sep} = \lim_{\nu\to 0} W_2^{sep} = \alpha^*[S - V(\overline{d}_2) - (1+\lambda)(\kappa(\overline{\theta}) - \overline{d}_2\overline{\theta})],$$

where

$$\overline{d}_2 = \frac{1}{2}\lim_{\nu\to 0} V'(\overline{d}_2) = \frac{1}{2}(1+\lambda)\overline{\theta} = \overline{d}^*;$$

$$L^{un} = \lim_{\nu\to 0} W_2^{un} = \alpha^*[S - V(d_2^s) - (1+\lambda)(\kappa(\overline{\theta}) - d_2^s\overline{\theta}) -$$

$$-\left(1 + \lambda - \frac{1}{\alpha^*}\right)(d_2^s\Delta\theta - \kappa(\overline{\theta}) + \kappa(\underline{\theta}))],$$

where

$$d_2^s = \frac{1}{2}\lim_{\nu\to 0} V'(d_2^s) = \frac{1}{2}\left[(1+\lambda)\underline{\theta} + \frac{\triangle\theta}{\alpha^*}\right] = \frac{1}{2}\underline{d}^* + \frac{\triangle\theta}{2\alpha^*}.$$

Consider two extreme cases.

First case. Let $\widetilde{K} < d_2^s$ but is sufficiently close to d_2^s for one may neglect the last terms in L^{un}. Then

$$L^{sep} - L^{un} = -\alpha^*\left(\frac{1}{2}(1+\lambda)\overline{\theta}\right)^2 + \alpha^*\left(\frac{1}{2}(1+\lambda)\underline{\theta} + \frac{1}{2}\frac{\triangle\theta}{\alpha^*}\right)^2 + \alpha^*\frac{1}{2}(1+\lambda)^2\overline{\theta}^2 -$$

$$-\alpha^*\frac{1}{2}(1+\lambda)^2\underline{\theta}\overline{\theta} - \alpha^*(1+\lambda)\frac{\overline{\theta}\triangle\theta}{2\alpha^*} = \frac{1}{4}(\triangle\theta)^2\alpha^*\left[\frac{1}{(\alpha^*)^2} - (1+\lambda)^2\right] > 0.$$

If ν is small and $\widetilde{K}$ is close to d_2^s and $\widetilde{K} < d_2^s$ then the majority-2 chooses a separating mechanism and, besides, the society's welfare function is equal to

$$W^{sep} = \frac{W_2^{sep}}{\alpha^*} + \nu\left(1 - \frac{1}{\alpha^*}\right)(\widetilde{K} - \overline{d}_2)\triangle\theta.$$

By using the pooling mechanism of "small" $\widetilde{K}$,

$$W^{un} = \frac{W_2^{un}}{\alpha^*} + (1-\nu)\left(1 - \frac{1}{\alpha^*}\right)(d_2^s - \widetilde{K})\triangle\theta.$$

Thus, $W^{sep} > W^{un}$, i.e. society's interests coincide with the choice of the interested sides.

Second case. Let $\widetilde{K} > \overline{d}_2$ but $\widetilde{K}$ be sufficiently close to $\overline{d}_2$ for the difference of the last term in L^{un} from $(d_2^s - \overline{d}_2)\triangle\theta$ might be neglected. Then

$$d_2^s - \widetilde{K} \to \frac{1}{2}\left[\frac{1}{\alpha^*} - (1+\lambda)\right]\triangle\theta$$

and therefore

$$\begin{aligned} L^{sep} - L^{un} = \frac{\Delta\theta^2}{4\alpha^*}&[(\alpha^*)^2(1+\lambda)^2 + 1 - 2\alpha^*(1+\lambda) - \\ &-2(1+\lambda)^2(\alpha^*)^2 + 4(1+\lambda)\alpha^* - 2] = \\ &= -\frac{(\triangle\theta)^2}{4\alpha^*}[(1+\lambda)\alpha^* - 1]^2 < 0. \end{aligned}$$

The majority-2 chooses the pooling mechanism if ν is small, and $\widetilde{K}$ is close to $\overline{d}_2$ and $\widetilde{K} > \overline{d}_2$.

Compare the social welfare functions:

$$\begin{aligned} W^{sep} - W^{un} = -\frac{(\triangle\theta)^2}{4(\alpha^*)^2}&[(1+\lambda)\alpha^* - 1]^2 + \\ +\frac{1}{2}\left(\frac{1}{\alpha^*} - 1\right)&\left[\frac{1}{\alpha^*} - (1+\lambda)\right](\Delta\theta)^2 > 0. \end{aligned}$$

The separating mechanism is preferable for the society. Pollution levels are linked by the following relation:

$$\underline{d}^* = \underline{d}_2 < \overline{d}_2 < d_2^s.$$

4. Comparison of separating and pooling mechanisms under small $\triangle\theta$

Now we will investigate the situation where: 1) $\triangle\theta$ is small; 2) the choice of the kind of the mechanism (separating or pooling) is made either by the society or by the regulator (the majority-1 or the majority-2); 3) in accordance to a kind of the mechanism, the regulator sets a menu of contracts or a uniform contract.

One can tract the value $\triangle\theta$ as a result of deviation from the point $\widehat{\theta} = \overline{\theta} = \underline{\theta}$ by decreasing the value $\underline{\theta}$ or by increasing the value $\overline{\theta}$. Notice that in the point $\widehat{\widehat{\theta}}$ the following equality holds:

$$\underline{d}_1 = \overline{d}^2 = d_1^s = d_2^s = \underline{d}^*.$$

We will suppose in this Section that the function $\kappa(\theta)$ is differentiable, then an approximate equality is valid:

$$\widetilde{K} \approx \kappa'(\widehat{\theta}).$$

Further an analysis of the type of the mechanism is conducted on the base of a comparison of the values of the objective functions and of their derivatives by use of the envelope theorem (e.g. (Takayama, 1994)).

It is easily seen that in the point $\widehat{\theta}$ the social welfare for the separating and the pooling mechanisms coincides as well as its first derivatives (they are given below).

4.1. Case of "small" $\widetilde{K}$ (the firm of type $\overline{\theta}$ receives a rent)

Lemma 1. *Let the case of "small" $\tilde{K}$ take place, the society chooses the mechanism, and the regulator defines only the menu of contracts, and*

$$B = p(\lambda + 2\nu - 1)(1+\lambda) + q\left(\lambda + 2\nu - 1 + \frac{1-\nu}{\alpha^*}\right)\left(1 + \lambda - \frac{1-\nu}{\alpha^*}\right).$$

Then:

1) under $B > 0$ the separating mechanism is preferable for the society,

2) under $B < 0$ the pooling mechanism is preferable for the society.

Proof. We will think about $\triangle\theta$ as about a result of decreasing $\underline{\theta}$ and compare mechanisms by second derivatives of the social welfare function in variable $\underline{\theta}$ in point $\widehat{\theta}$. These derivatives are calculated in the following parts I and II of the proof, and then in part III the comparison of the separating and the pooling mechanisms is made.

I. The separating mechanism.

I.i. The majority-1 is in power. The social welfare equals

$$W(\underline{\theta}) = \frac{W^1(\underline{\theta})}{\alpha^*} + (1-\nu)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \triangle\theta \underline{d}_1),$$

where $W^1(\underline{\theta})$ is the objective function of the majority-1 described by (7); the pollution levels $\underline{d}_1, \overline{d}^*$ are found in point 3.1.1. Applying to $W^1(\underline{\theta})$ the envelope theorem, we find

$$\frac{dW(\underline{\theta})}{d\underline{\theta}} = -(\lambda+\nu)(\kappa'(\underline{\theta}) - \underline{d}_1) + (1-\nu)\triangle\theta\frac{d\underline{d}_1}{d\underline{\theta}}.$$

It follows from (8) that

$$\frac{d\underline{d}_1}{d\underline{\theta}} = \frac{1+\lambda}{\nu V''(\underline{d}_1)}$$

$$\frac{d^2\underline{d}_1}{d\underline{\theta}^2} = -\frac{(1+\lambda)^2 V'''(\underline{d}_1)}{\nu^2[V''(\underline{d}_1)]^3}.$$

We find the second derivative of the social welfare function in point $\widehat{\theta}$:

$$\frac{d^2W(\underline{\theta})}{d\underline{\theta}^2}|_{\underline{\theta}=\widehat{\theta}} = -(\lambda+\nu)\kappa''(\widehat{\theta}) + \frac{(\lambda+2\nu-1)(1+\lambda)}{\nu V''(\underline{d}^*)}.$$

I.ii. The majority-2 is in power. The social welfare is

$$W(\underline{\theta}) = \frac{W^2(\underline{\theta})}{\alpha^*} + (1-\nu)\left(1 - \frac{1}{\alpha^*}\right)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \triangle\theta\underline{d}_2),$$

where $W^2(\underline{\theta})$ is the objective function of the majority-2 described by (10), $\underline{d}_2, \overline{d}^*$ are the pollution levels defined by the majority-2 (they are found in 3.1.2). Applying the envelope theorem to $W^2(\underline{\theta})$ we find

$$\frac{dW(\underline{\theta})}{d\underline{\theta}} = -(\lambda+\nu)(\kappa'(\underline{\theta}) - \underline{d}_2) + (1-\nu)\left(1 - \frac{1}{\alpha^*}\right)\triangle\theta\frac{d\underline{d}_2}{d\underline{\theta}}.$$

It follows from (11), that

$$\frac{d\underline{d}_2}{d\underline{\theta}} = \frac{1+\lambda-\frac{1-\nu}{\alpha^*}}{\nu V''(\underline{d}_2)}$$

$$\frac{d^2\underline{d}_2}{d\underline{\theta}^2} = -\frac{V'''(\underline{d}_2)}{[V''(\underline{d}_2)]^3}\left(\frac{1+\lambda-\frac{1-\nu}{\alpha^*}}{\nu}\right)^2.$$

In result in point $\widehat{\theta}$:

$$\frac{d^2W(\underline{\theta})}{d\underline{\theta}^2}|_{\underline{\theta}=\widehat{\theta}} = -(\lambda+\nu)\kappa''(\widehat{\theta}) + \left[\lambda+\nu-(1-\nu)(1-\frac{1}{\alpha^*})\right]\frac{1+\lambda-\frac{1-\nu}{\alpha^*}}{\nu V''(\underline{d}^*)}.$$

II. The pooling mechanism.

II.i. The majority-1 is in power. The social welfare equals

$$W(\underline{\theta}) = \frac{W^1(\underline{\theta})}{\alpha^*} + (1-\nu)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \triangle\theta d_1^s),$$

where $W^1(\underline{\theta})$ is the objective function of the majority-1, described by (13). The pollution level d_1^s, as is shown in subsection 3.2.1, equals $\underline{d}^*$. Analogously to the case of the separating mechanism, in point $\widehat{\theta}$,

$$\frac{d^2W(\underline{\theta})}{d\underline{\theta}^2}|_{\underline{\theta}=\widehat{\theta}} = -(\lambda+\nu)\kappa''(\widehat{\theta}) + \frac{(\lambda+2\nu-1)(1+\lambda)}{V''(\underline{d}^*)}.$$

II.ii. Majority-2 is in power. Society's welfare equals

$$W(\underline{\theta}) = \frac{W^2(\underline{\theta})}{\alpha^*} + (1-\nu)\left(1-\frac{1}{\alpha^*}\right)(\kappa(\underline{\theta}) - \kappa(\overline{\theta}) + \triangle\theta d_2^s),$$

where $W^2(\underline{\theta})$ is the objective function of the majority-2, described by equality (14). Pollution level d_2^s is defined by equation (15). Similarly to the case of separating mechanism, in point $\widehat{\theta}$,

$$\frac{d^2W(\underline{\theta})}{d\underline{\theta}^2}|_{\underline{\theta}=\widehat{\theta}} = -(\lambda+\nu)\kappa''(\widehat{\theta}) + \left[\lambda+\nu-(1-\nu)\left(1-\frac{1}{\alpha^*}\right)\right]\frac{1+\lambda-\frac{1-\nu}{\alpha^*}}{V''(\underline{d}^*)}.$$

III. Comparison of the separating and the pooling mechanisms in point $\hat{\theta}$.

The expected value of the second derivative of the social welfare under the separating mechanism is

$$D^{sep} = -(\lambda+\nu)\kappa''(\hat{\theta}) + \frac{B}{2\nu V''(\underline{d}^*)},$$

and under the pooling mechanism is

$$D^{un} = -(\lambda+\nu)\kappa''(\hat{\theta}) + \frac{B}{2V''(\underline{d}^*)}.$$

Preferable, from the point of view of the social welfare, is the mechanism with a greater value of the second derivative in $\underline{\theta}$. Let us stress that the sign of $\Delta\theta$ (in this case it is negative) doesn't matter. Actually,

$$W(\underline{\theta}) = \frac{W''(\hat{\theta})}{2}(\Delta\theta)^2 + o((\Delta\theta)^2).$$

If $B > 0$ then $\frac{B}{V''(\hat{d})} < \frac{B}{\nu V''(\hat{d})}$ and the separating mechanism is preferable. If $B < 0$ then the pooling mechanism is preferable.

For us the most interesting case is the typical for the most of developing and transition economies one, when with a large probability the majority-2 is in power and "the share" of firm of type $\bar{\theta}$ is great, i.e. q is great, and condition (16) holds.

Theorem 2. *Let the case of "small" $\tilde{K}$ be considered and let $\Delta\theta$ be small. Under typical for developing and transition economies conditions, when condition (16) holds and q is sufficiency large, if the choice of type of mechanism is made by the society then the pooling mechanism will be chosen. If the majority-2 is in power and chooses both a mechanism and a menu of contracts then also the pooling mechanism will be chosen .*

Proof. In case under consideration, a sign of the magnitude B is defined by the second term which is negative. By Lemma 1, the unifying mechanism is preferable for the society.

Under condition (16) the condition of feasibility of the separating mechanism is violated, therefore the majority-2 will choose the pooling mechanism.

Theorem 3. *Let the case of "small" $\tilde{K}$ be considered and let $\Delta\theta$ be small. Under conditions when the "share" of the firms of type $\bar{\theta}$ in the economy is large (the condition (16) holds) and the majority-1 is in power this regulator chooses the separating mechanism, this coincides with the society's interests only if $\lambda + 2\nu - 1 > 0$. If $\lambda + 2\nu - 1 < 0$ then the unifying mechanism is preferable for the society. The pollution levels are linked by the relation:*

$$\underline{d}_1 < d_1^s = \underline{d}^* < \bar{d}^*.$$

Proof. In case of the separating mechanism the objective function of the majority-1 $W^1(\underline{\theta})$ is defined by Equation (7), and the pollution level $\underline{d}_1$ by Equation (8). In case of the pooling mechanism the objective function $W^1(\underline{\theta})$ is defined by Equation (13), the pollution level d_1^s equals $\underline{d}^*$. In point $\hat{\theta}$ the values of these functions coincide, the pollution levels coincide and equal $\underline{d}^*$, the first derivatives coincide and equal to

$$-\alpha^*(1+\lambda)(\kappa'(\underline{\theta}) - \underline{d}^*).$$

The second derivatives in these two cases are equal, correspondingly :

$$D^{sep} = \alpha^*(1+\lambda)\left[\frac{1+\lambda}{\nu V''(\underline{d}^*)} - \kappa''(\hat{\theta})\right],$$

$$D^{un} = \alpha^*(1+\lambda)\left[\frac{1+\lambda}{V''(\underline{d}^*)} - \kappa''(\hat{\theta})\right].$$

Thus, the majority-1 chooses the separating mechanism. If $\lambda + 2\nu - 1 > 1$, then, as is seen from Lemma 1, the society in whole would choose the separating mechanism, and if $\lambda + 2\nu - 1 < 1$ then the pooling mechanism.

4.2. The case of "large" $\tilde{K}$ (a rent is obtained by the firm of type $\underline{\theta}$)

As has been already said, the case of "large" $\tilde{K}$ seems to be typical for developed countries.

Lemma 2. *Suppose, that the society defines the type of the mechanism, and the regulator chooses the menu of contracts. The case of "large" $\tilde{K}$ is under consideration. Let the majority-2 be in power with a large probability (more exactly, p is so small that the sign of the magnitude*

$$C = p(1+\lambda)(1+\lambda-2\nu) + q\left[1+\lambda+-2\nu+\frac{\nu}{\alpha^*}\right]\left(1+\lambda-\frac{\nu}{\alpha^*}\right).$$

is defined by the second term), and the "the share" of firms of type $\bar{\theta}$ is sufficiently large (inequality $1+\lambda > \nu/\alpha^*$ holds). Then $C > 0$ and the separating mechanism is preferable for the society. If, under the same conditions, $1+\lambda < \nu/\alpha^*$ then the pooling mechanism is preferable for the society.

Proof. For "large" $\tilde{K}$ it is convenient, as it was done in (Laffont, 2000), to tract value $\Delta\theta$ as a result of increasing the magnitude $\bar{\theta}$. In parts I and II of the proof we will obtain an expression for the derivatives of the social welfare functions in $\bar{\theta}$ in point $\hat{\theta}$, and then in part III of the proof we will execute a direct comparison of the separating and the pooling mechanisms.

I. Separating mechanism.

I.i. The majority-1 is in power. The social welfare equals

$$W(\bar{\theta}) = \frac{W^1(\bar{\theta})}{\alpha^*} + \nu(\tilde{K} - \bar{d}_1)\Delta\theta = \frac{W^1(\bar{\theta})}{\alpha^*} + \nu(\kappa(\bar{\theta}) - \kappa(\underline{\theta}) - \bar{d}_1\Delta\theta),$$

where the objective function of the majority-1 $W^1(\bar{\theta})$ has the form:

$$W^1(\bar{\theta}) = \alpha^* E[S - V(d) - (1+\lambda)(\kappa(\theta) - \theta d) - (1+\lambda)U].$$

The pollution levels $\underline{d}^*$ and $\bar{d}_1$ are defined, correspondingly, by equations (9) and (2). Applying the envelope theorem to $W^1(\bar{\theta})$ we find:

$$\frac{dW^1(\bar{\theta})}{d\bar{\theta}} = \alpha^*[-\nu(1+\lambda)(\kappa'(\bar{\theta}) - \bar{d}_1) - (1-\nu)(1+\lambda)(\kappa'(\bar{\theta}) - \bar{d}_1)]$$

$$= -\alpha^*(1+\lambda)(\kappa'(\bar{\theta} - \bar{d}_1)$$

From (9) we find:

$$\frac{d\bar{d}_1}{d\bar{\theta}} = \frac{1+\lambda}{(1-\nu)V''(\bar{d}_1)},$$

$$\frac{d^2\bar{d}_1}{d\bar{\theta}^2} = -\frac{V'''(\bar{d}_1)}{V''(\bar{d}_2)}\left(\frac{d\bar{d}_1}{d\bar{\theta}}\right)^2 = -\frac{(1+\lambda)^2 V'''(\bar{d}_1)}{(1-\nu)^2[V''(\bar{d}_1)]^3}.$$

Applying the envelope theorem once more, we obtain

$$\frac{d^2W^1(\bar{\theta})}{d\bar{\theta}^2} = -\alpha^*(1+\lambda)\left(\kappa''(\bar{\theta}) - \frac{d\bar{d}_1}{d\bar{\theta}}\right).$$

Thus, for the social welfare function we have:

$$\frac{dW(\bar{\theta})}{d\bar{\theta}} = (\nu - 1 - \lambda)\kappa''(\bar{\theta}) + \frac{(1+\lambda-2\nu)(1+\lambda)}{(1-\nu)V''(\bar{d}_1)} + \frac{\mu(1+\lambda)^2 V'''(\bar{d}_1)}{(1-\nu)^2[V''(\bar{d}_1)]^3}\Delta\theta.$$

I.ii. The majority-2 is in power. The social welfare equals

$$W(\bar{\theta}) = \frac{W^2(\bar{\theta})}{\alpha^*} + \nu\left(1 - \frac{1}{\alpha^*}\right)(\tilde{K} - \bar{d}_2)\Delta\theta =$$

$$= \frac{W^2(\bar{\theta})}{\alpha^*} + \nu\left(1 - \frac{1}{\alpha^*}\right)(\kappa(\bar{\theta}) - \kappa(\underline{\theta}) - \bar{d}_2\Delta\theta),$$

where the objective function of the majority-2 $W^2(\bar{\theta})$ has the form

$$W^2(\bar{\theta}) = \alpha^* E\left[S - V(d) - (1+\lambda)(\kappa(\theta) - \theta d) - \left(1 + \lambda - \frac{1}{\alpha^*}\right)U\right].$$

The pollution levels $\underline{d}^*$ and $\bar{d}_2$ are defined, correspondingly, by relations (2) and (12). Applying the envelope theorem to $W^2(\bar{\theta})$ we find:

$$\frac{dW^2(\bar{\theta})}{d\bar{\theta}} = \alpha^*\left[-(1+\lambda)(\kappa'(\bar{\theta}) - \bar{d}_2) + \frac{\nu}{\alpha^*}(\kappa'(\bar{\theta}) - \bar{d}_2)\right].$$

From (12) we find:

$$\frac{d\bar{d}_2}{d\bar{\theta}} = \frac{1+\lambda-\nu/\alpha^*}{(1-\nu)V'''(\bar{d}_2)},$$

$$\frac{d^2\bar{d}_2}{d\bar{\theta}^2} = -\frac{(1+\lambda-\nu/\alpha^*)^2\, V'''(\bar{d}_2)}{(1-\nu)[V''(\bar{d}_2)]^3}.$$

Applying the envelope theorem once more, we obtain:

$$\frac{d^2W^2(\bar{\theta})}{d\bar{\theta}^2} = \alpha^*\left[\left(\frac{\nu}{\alpha^*} - (1+\lambda)\right)\kappa''(\bar{\theta}) + \frac{(\nu/\alpha^* - (1+\lambda))^2}{(1-\nu)V''(\bar{d}_2)}\right]$$

Thus, we have for the social welfare function:

$$\frac{d^2W^2(\bar{\theta})}{d\bar{\theta}} =$$

$$= (\nu - 1 - \lambda)\kappa''(\bar{\theta}) + \left(1 + \lambda - 2\nu + \frac{\nu}{\alpha^*}\right)\frac{1+\lambda-\nu/\alpha^*}{(1-\nu)V''(\bar{d}_2)} +$$

$$+\nu\left(1 - \frac{1}{\alpha^*}\right)\frac{(1+\lambda-\nu/\alpha^*)^2 V'''(\bar{d}_2)}{(1-\nu)^2[V''(\bar{d}_2)]^3}\Delta\theta.$$

II. The unifying mechanism.

II.i. The majority-1 is in power. The social welfare equals

$$W(\bar{\theta}) = \frac{W^1(\bar{\theta})}{\alpha^*} + \nu(\kappa(\bar{\theta}) - \kappa(\underline{\theta}) - d_1^h\Delta\theta),$$

where the objective function of the majority-1 $W^1(\bar{\theta})$ has the form:

$$W^1(\bar{\theta}) = \alpha^*[S - V(d_1^h) - (1+\lambda)(\nu(\kappa(\underline{\theta}) - \underline{\theta}d_1^h) + (1-\nu)(\kappa(\bar{\theta}) - \bar{\theta}d_1^h)) -$$

$$-\nu(1+\lambda)(\kappa(\bar{\theta}) - \kappa(\underline{\theta}) - d_1^h \Delta\theta)].$$

The pollution level d_1^h is defined by the equation

$$V'(d_1^h) = (1+\lambda)(\nu\underline{\theta} + (1-\nu)\bar{\theta}) + \nu(1+\lambda)\Delta\theta = (1+\lambda)\bar{\theta}. \qquad (17)$$

Hence, $d_1^h = \bar{d}^*$. Applying to $W^1(\bar{\theta})$ the envelope theorem, we find:

$$\frac{dW^1(\bar{\theta})}{d\bar{\theta}} = \alpha^*[-(1+\lambda)(1-\nu)(\kappa'(\bar{\theta}) - d_1^h) - (1+\lambda)\nu(\kappa'(\bar{\theta}) - d_1^h)] =$$

$$= -\alpha^*(1+\lambda)(\kappa'(\bar{\theta}) - d_1^h).$$

From (17) we obtain:

$$\frac{dd_1^h}{d\bar{\theta}} = \frac{1+\lambda}{V''(d_1^h)},$$

$$\frac{d^2 d_1^h}{d\bar{\theta}^2} = -\frac{V'''(d_1^h)}{V''(d_1^h)}\left(\frac{dd_1^h}{d\bar{\theta}}\right)^2 = -\frac{(1+\lambda)^2 V'''(d_1^h)}{[V''(d_1^h)]^3}.$$

Applying the envelope theorem once more we obtain:

$$\frac{d^2W^1(\bar{\theta})}{d\bar{\theta}^2} = -\alpha^*(1+\lambda)\left(\kappa''(\bar{\theta}) - \frac{dd_1^h}{d\bar{\theta}}\right) = -\alpha^*(1+\lambda)\left(\kappa''(\bar{\theta}) - \frac{1+\lambda}{V''(d_1^h)}\right).$$

Thus, for the social welfare function:

$$\frac{dW(\bar{\theta})}{d\bar{\theta}} = -(1+\lambda)(\kappa'(\bar{\theta}) - d_1^h) + \nu(\kappa'(\bar{\theta}) - d_1^h) - \nu\frac{dd_1^h}{d\bar{\theta}}\Delta\theta,$$

$$\frac{d^2W(\bar{\theta})}{d\bar{\theta}^2} = (\nu - 1 - \lambda)\kappa''(\bar{\theta}) + (1+\lambda-2\nu)\frac{1+\lambda}{V''(d_1^h)} + \nu\frac{(1+\lambda)^2 V'''(d_1^h)}{[V''(d_1^h)]^3}\Delta\theta.$$

II.ii. Majority-2 is in power. Social welfare equals

$$W(\bar{\theta}) = \frac{W^2(\bar{\theta})}{\alpha^*} + \nu\left(1 - \frac{1}{\alpha^*}\right)(\kappa(\bar{\theta}) - \kappa(\underline{\theta}) - d_2^h\Delta\theta),$$

where the objective function of the majority-2:

$$W^2(\bar{\theta}) = \alpha^*[S - V(d_2^h) - (1+\lambda)(\nu(\kappa(\underline{\theta}) - \underline{\theta}d_2^h) + (1-\nu)(\kappa(\bar{\theta}) - \bar{\theta}d_2^h)) -$$

$$-\nu\left(1+\lambda - \frac{1}{\alpha^*}\right)(\kappa(\bar{\theta}) - \kappa(\underline{\theta}) - d_2^h\Delta\theta)].$$

The pollution level d_2^h is defined from equation:

$$V'(d_2^h) = (1+\lambda)(\nu\underline{\theta} + (1-\nu)\bar{\theta}) + \nu\left(1+\lambda - \frac{1}{\alpha^*}\right)\Delta\theta = (1+\lambda)\bar{\theta} - \frac{\nu}{\alpha^*}\Delta\theta. \qquad (18)$$

Applying to $W^2(\bar{\theta})$ the envelope theorem we find:

$$\frac{dW^2(\bar{\theta})}{d\bar{\theta}} = \alpha^* \left[-(1+\lambda)(\kappa'(\bar{\theta}) - d_2^h) + \frac{\nu}{\alpha^*}(\kappa'(\bar{\theta}) - d_2^h) \right].$$

From (18) we obtain:

$$\frac{dd_2^h}{d\bar{\theta}} = \frac{1+\lambda-\nu/\alpha^*}{V''(d_2^h)},$$

$$\frac{d^2 d_2^h}{d\bar{\theta}^2} = -\frac{(1+\lambda-\nu/\alpha^*)^2 V'''(d_2^h)}{(1-\nu)[V''(d_2^h)]^3}.$$

Applying the envelope theorem once more, we obtain:

$$\frac{d^2W^2(\bar{\theta})}{d\bar{\theta}^2} = \alpha^* \left[\left(\frac{\nu}{\alpha^*} - (1+\lambda) \right) \kappa''(\bar{\theta}) + \frac{(\nu/\alpha^* - (1+\lambda))^2}{V''(d_2^h)} \right].$$

Thus, for the social welfare function we have:

$$\frac{d^2W(\bar{\theta})}{d\bar{\theta}^2} = (\nu - 1 - \lambda)\kappa''(\bar{\theta}) + \left(1+\lambda-2\nu+\frac{\nu}{\alpha^*}\right)\frac{1+\lambda-\nu/\alpha^*}{V''(d_2^h)} +$$

$$+\nu\left(1-\frac{1}{\alpha^*}\right)\frac{(1+\lambda-\nu/\alpha^*)^2 V'''(d_2^h)}{[V''(d_2^h)]^3}\Delta\theta.$$

III. Comparison of the separating and the pooling mechanisms in point $\hat{\theta}$.

In the point $\hat{\theta}$ (when $\Delta\theta$=0) the values of the social welfare function for the separating and the pooling mechanisms coincide, the pollution levels $\bar{d}_2$ and d_2^h coincide and equal $\underline{d}^*$, the first derivatives also coincide. The expected value of the second derivative of the social welfare under the separating mechanism is equal to:

$$D^{sep} = (\nu - 1 - \lambda)\kappa''(\hat{\theta}) + \frac{C}{(1-\nu)V''(\underline{d}^*)},$$

and under the unifying mechanism:

$$D^{un} = (\nu - 1 - \lambda)\kappa''(\bar{\theta}) +$$

$$+\left[p(1+\lambda-2\nu)(1+\lambda) + q\left[\lambda+1-\nu-\nu\left(1-\frac{1}{\alpha^*}\right)\right]\left(1+\lambda-\frac{\nu}{\alpha^*}\right)\right]\frac{1}{V''(\underline{d}^*)} =$$

$$= (\nu - 1 - \lambda)\kappa''(\hat{\theta}) + \frac{C}{V''(\underline{d}^*)}.$$

It is clear, that

$$\lambda + 1 - \nu - \nu\left(1 - \frac{1}{\alpha^*}\right) > \lambda + 1 - \nu > 0,$$

so the sign of C under small p is defined by the sign of $1+\lambda-\nu/\alpha^*$. If $1+\lambda > \nu/\alpha^*$ then $C > 0$, then the separating mechanism is preferable for the society. If $1+\lambda < \nu/\alpha^*$ then $C < 0$ and the unifying mechanism is preferable for the society.

Theorem 4. *In the case of "large" $\widetilde{K}$ if the majority-2 is in power and chooses both the mechanism and the menu of contracts, and the condition $\nu/\alpha^* < 1 + \lambda$ (condition of feasibility of the separating mechanism) holds, then the majority-2 chooses the separating mechanism. Besides, the pollution levels are linked by the following relation:*

$$\underline{d}^* < d_2^h < \bar{d}_2.$$

If $\nu/\alpha^* > 1 + \lambda$ then only the pooling mechanism is available.

Proof. In the point $\hat{\theta}$ (where $\Delta\theta = 0$) the values of the objective function of the majority-2 for the separating and the pooling mechanisms coincide, the pollution levels $\bar{d}_2$ and d_2^h coincide and equal $\underline{d}^*$, the first derivatives also coincide. The second derivatives in these two cases are equal, correspondingly to:

$$D^{sep} = \alpha^* \left[\left(\frac{\nu}{\alpha^*} - (1+\lambda) \right) \kappa''(\bar{\theta}) + \frac{(\nu/\alpha^* - (1+\lambda))^2}{(1-\nu)V''(\underline{d}^*)} \right] =$$

$$= (\nu - \alpha^*(1+\lambda))\kappa''(\hat{\theta}) + \frac{\alpha^* V''(\underline{d}^*) G^2}{1-\nu}$$

and

$$D^{un} = (\nu - \alpha^*(1+\lambda))\kappa''(\hat{\theta}) + \alpha^* V''(\underline{d}^*) G^2,$$

where

$$G = \frac{\nu/\alpha^* - (1+\lambda)}{V''(\underline{d}^*)}.$$

Thus, independently on relation between values ν/α^* and $1+\lambda$, the majority-2 will choose the separating mechanism.

Theorem 5. *In the case of "large" $\tilde{K}$, if the majority-1 is in power and chooses both the mechanism and the menu of contracts then it will choose the separating mechanism.*

Proof. In the point $\hat{\theta}$ ($\Delta\theta = 0$) the values of the objective function of the majority-1 for the separating and the pooling mechanisms coincide, the pollution levels $\bar{d}_1$ and d_1^h coincide and equal $\underline{d}^*$, the first derivatives coincide. The second derivatives in these two cases are equal:

$$D^{sep} = -\alpha^*(1+\lambda) \left(\kappa''(\hat{\theta}) - \frac{1+\lambda}{(1-\nu)V''(\underline{d}^*)} \right)$$

and

$$D^{un} = -\alpha^*(1+\lambda) \left(\kappa''(\hat{\theta}) - \frac{1+\lambda}{V''(\underline{d}^*)} \right).$$

Thus, the majority-1 will choose the separating mechanism.

4.3. Discussion of results

The results of research are brought in Tables 1 and 2.

Table 1 corresponds to the case which seems to be typical for many developing and transition economies: "dirty" firms are relatively effective and their share in

Table1: The choice of the kind of the mechanism and the pollution level under "small" $\tilde{K}$ and small ν (by $\nu < 1 - (1+\lambda)\alpha^*$)

Who sets mechanism	Who sets contracts menu	Admissible pollution levels	What a mechanism is chosen
Society	Majority-1	$\underline{d}^*$	Pooling
	Majority-2	d_2^s	
Majority-2	Majority-2	d_2^s	
Majority-1	Majority-1	$\underline{d}_1$ $\overline{d}^*$	Separating

Table2: Choice of kind of mechanism and pollution levels by "large" $\tilde{K}$

Who sets the mechanism	Who sets contracts menu	Admissible pollution levels	What a mechanism is chosen
Society	Majority-1	$\overline{d}^*$, if $\nu > \frac{1+\lambda}{2}$	Pooling
		$\underline{d}^*$ $\overline{d}^1$, if $\nu < \frac{1+\lambda}{2}$	Separating
Society or Majority-2	Majority-2	d_2^h, if $\nu > (1+\lambda)\alpha^*$	Pooling
		$\underline{d}^*$ and $\overline{d}_2$, if $\nu < (1+\lambda)\alpha^*$	Separating
Majority-1	Majority-1	$\underline{d}^*$ $\overline{d}^1$	Separating

the economy, $1-\nu$, is relatively large. Moreover, feasible pollution levels are linked by the relation:

$$\underline{d}_1 < \underline{d}^* < \bar{d}^* < d_2^s,$$

Table 2 corresponds the case typical for developed country where "green" firms are relatively efficient. In this case the following relations between admissible pollution levels hold:

If $\nu > (1+\lambda)\alpha^*$ then $d_2^h < \underline{d}^* < \bar{d}^* < \bar{d}_1$.

If $\nu < (1+\lambda)\alpha^* < 1$ then $\underline{d}^* < d_2^h < \bar{d}_2 < \bar{d}^* < \bar{d}_1$.

If $\nu < 1 < (1+\lambda)\alpha^*$ then $\underline{d}^* < d_2^h < \bar{d}^* < \bar{d}_2 < \bar{d}_1$.

Notice that in all cases considered in Table 2 the share of "green" firms ν may be either higher or wittingly higher than in the cases considered in Table 1. The situations represented in Table 1 and Table 2, in our opinion, quite correspond to economic conditions in developing and transition economies and in developed countries, correspondingly.

Comparing the right parts of the tables, we see, that the employment of the separating (market) mechanism may be expected in a more degree in developed countries than in developing and transition economies.

In the case typical for developing and transition economies (Table 1) the greater pollution level d_2^s of "green" firms is reached under the pooling mechanism when the interested majority sets the menu of contracts.

On the contrary, in the case typical for developed countries (Table 2) the majority-2 appears to be the most effective ecological regulator.

Table 2, however, allows to make another conclusion: when the share of "green" firms in the economy increases, one may expect in developed countries a higher degree of pooling mechanism employment.

5. Conclusion

In this paper on base of the contracts theory the work of an ecological policy mechanism is studied under different conditions, including both economic components (economic efficiency of different types of firms and their "share" (frequency) in the economy) and a political component (who namely – the society or the regulator – makes decision concerning the type of the mechanism – pooling or separating, who is in power and makes decision about admissible pollution levels). Analysis shows that under the same frame mechanism its variety and the resulting economical policy depend considerably on these conditions.

Thus, the research put under doubt a broadly spreading view about a possibility of an adequate transfer into an arbitrary taken transition or developing economy of the institutions which have proved themselves perfect in one or another developed country.

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Equilibrium Uniqueness Results for Cournot Oligopolies Revisited

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Abstract We revisit and compare equilibrium uniqueness results for homogeneous Cournot oligopolies. In doing this we provide various useful and interesting results for which it is difficult to give appropriate reference in the literature. We also propose problems for further research.

Keywords: Aggregative game, equilibrium (semi-)uniqueness, Fisher-Hahn conditions, marginal reduction, oligopoly.

1. Introduction

Cournot equilibrium is one of most studied equilibrium concepts in economics and in game theory. A variety of results have been obtained in the analysis of the set of Cournot equilibria for homogeneous good Cournot oligopolies; many of these results concern existence, uniqueness and stability of Cournot equilibria. In this article we are mostly interested in Cournot equilibrium uniqueness results and related issues. Henceforth, unless otherwise specified, by an oligopoly we mean a homogeneous good Cournot oligopoly, and by an equilibrium a Cournot equilibrium.

To the best of our knowledge, in the literature, all equilibrium uniqueness results concerning 'unspecific' oligopolies with continuous profit functions and an indefinite number of possibly non-identical firms assume the quasi-concavity of conditional[1] profit functions (see also Vives(2001) for an overview of the literature). When each firm's profit function is continuous and quasi-concave in the firm's strategic variable, the problem of the existence of an equilibrium follows easily from the Nikaido-Isoda theorem for games in strategic form (or, though less directly, from Kakutani fixed point theorem) in case strategy sets are compact. But when compacteness of strategy sets is not assumed some other 'compactness' condition must be imposed on profit functions. We shall provide, in Theorem 1, a useful theorem that can be employed in general formulations of such a 'compactness' condition on profit functions; one of these formulations is contained, e.g., in our Theorem 5.

So, in light of Theorem 1 (and Theorem 5), the difficult part of the equilibrium uniqueness results appeared in the literature for 'unspecific' oligopolies in fact concerns only equilibrium semi-uniqueness, i.e., the existence at most one equilibrium.

[1] The conditional profit function of a firm, for a given production of the opponents, is the profit function of the firm as a function of only its production. The terminology and the setting will be fixed next in Section 2.

One of the aims of our article is to give a review of equilibrium uniqueness results for classes of 'unspecific' oligopolies. Excluding special results like those for duopolies and symmetric oligopolies, we identify in the literature three important theorems that, with some goodwill, imply all other equilibrium uniqueness results under consideration.[2] These theorems are Theorems 7–10 below. Theorem 7 concerns an equilibrium semi-uniqueness result by Murphy et al.(1982)Murphy, Sherali, and Soyster for concave aggregate revenue and convex cost functions, Theorem 9 concerns an equilibrium semi-uniqueness result by Gaudet and Salant(1991) formulated in terms of two Fisher-Hahn conditions, and Theorem 10 concerns a 'special' equilibrium uniqueness result by Szidarovszky and Okuguchi(1997) that deals with the price function $p(y) = y^{-1}$ (however, it is good to note that in this last 'special' equilibrium uniqueness result profit functions are not continuous and equilibrium existence is an issue).

Many equilibrium uniqueness (and existence) results possess generalizations which are seldom provided in the literature; perhaps, the main reason for this is that these generalizations require technicalities which often partially hide an author's main contribution. On the other hand these generalizations sometimes appear in the literature. One of the aims of this paper is also to provide some formal statements (and proofs) of some of these 'unesthetical', albeit interesting, generalizations. By showing this type of results we facilitate, in our opinion, the discernment of methological novelties that have nothing to do with these 'unesthetical' generalizations. It is good to remark that through the paper we shall generally assume the differentiability of cost functions. Many results of this paper continue to hold when this condition is dispensed with, but generalizations of this kind requires modifications in the proofs and do not belong to the 'unesthetical' generalizations considered in this article.

The article is organised as follows. Section 3 deals with some results for a class of aggregative games in strategic form; the setting of Section 3 refers more generally to aggregative games and, in particular, to oligopolies. Section 4 deals with general results for oligopolies and Section 5 with the three above-mentioned theorems.

2. Setting

2.1. Games in Strategic Form

We deal with n player games in strategic form where $N := \{1, \ldots, n\}$ is the set of players and for all $i \in N$, X_i is player i's strategy set and f_i is player i's payoff function. Henceforth, X_i is non-empty and

$$f_i : \mathbf{X} \to \mathbb{R}$$

where $\mathbf{X} := X_1 \times \cdots \times X_n$.

For $i \in N$, let $\mathbf{X}_{\hat{\imath}} := X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n$. We sometimes identify $\mathbf{X}$ with $X_i \times \mathbf{X}_{\hat{\imath}}$ and then write $\mathbf{x} \in \mathbf{X}$ as $\mathbf{x} = (x_i; \mathbf{x}_{\hat{\imath}})$. For $i \in N$ and $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$, the *conditional payoff function* $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ is defined by $f_i^{(\mathbf{z})}(x_i) := f_i(x_i; \mathbf{z})$. $\mathbf{x} \in \mathbf{X}$ is a *(Nash) equilibrium* if for all $i \in N$, x_i is a maximiser of the function $f_i^{(\mathbf{x}_{\hat{\imath}})}$. By

E

[2] We shall not consider also (still) unpublished works like Quartieri(2008) and Ewerhart(2011).

we denote the set of equilibria. If the game is Γ, then we also denote this set by $E(\Gamma)$. For $i \in N$, the correspondence $R_i : \mathbf{X}_{\hat{\imath}} \multimap X_i$ is defined by

$$R_i(\mathbf{z}) \ := \ \operatorname{argmax} f_i^{(\mathbf{z})}.$$

R_i is called the *best reply correspondence* of player i. From now on we always assume, if not stated otherwise, that for each player $i \in N$!

$$X_i = \mathbb{R}_+, \text{ or } X_i = [0, m_i] \text{ where } m_i > 0.$$

2.2. Oligopolies

In this article we understand by a *(homogeneous Cournot) oligopoly* a game in strategic form where the payoff function (also called *profit function*) of player (also called *firm*) i is given by

$$f_i(\mathbf{x}) = p(\sum_{l=1}^{n} x_l)x_i - c_i(x_i).$$

Here $c_i : X_i \to \mathbb{R}$ is called the *cost function* of firm i and $p : Y \to \mathbb{R}$ is called the *price function* (or *inverse demand function*); the domain Y of p is the Minkowsi-sum $Y \ := \ \sum_{l=1}^{n} X_l$. In case $X_i = [0, m_i]$, we say that firm i has a *capacity constraint*.[3]

The *aggregate revenue function* $r : Y \to \mathbb{R}$ is defined by

$$r(y) \ := \ p(y)y.$$

A Nash equilibrium of an oligopoly is also referred to as a *Cournot equilibrium*.

It is good to note that the value of p at 0 is not important, as the profit functions do not depend on the value of p at 0.[4]

If there exists

$$v \in Y \cup \{+\infty\}$$

such that $p(y) > 0$ $(0 < y < v)$ and $p(y) \leq 0$ $(y > v)$, then such v is unique. In this case we refer to it as the *market satiation point* of p. Note $v = +\infty \ \Leftrightarrow \ p(y) > 0$ $(y \in Y \setminus \{0\})$. Also note that each price function p that is decreasing on $Y \setminus \{0\}$ has a market satiation point.

3. Games in Strategic Form

3.1. Marginal Reductions

Consider a game in strategic form. By a *linear co-strategy function* we mean a function $q : \mathbf{X} \to \mathbb{R}$ of the form $q(\mathbf{x}) \ := \ \sum_{l=1}^{n} q_l x_l$ with the q_l positive. The linear co-strategy function given by $\mathbf{x} \mapsto \sum_{l=1}^{n} x_l$ is denoted by α. Given a linear co-strategy function, we write $Y_q \ := \ q(\mathbf{X})$; also we write Y instead of Y_α . Finally, we also also write $\underline{\mathbf{z}} = \sum_{l=1}^{m} z_l$ for $\mathbf{z} \in \mathbb{R}^m$.

Let $i \in N$. Any pair $(t_i; q)$ where q is a linear co-strategy function and

$$t_i : X_i \times Y_q \to \mathbb{R},$$

---

[3] So if at least one firm does not have a capacity constraint, then $Y = \mathbb{R}_+$ and otherwise $Y = [0, \sum_{l=1}^{n} m_l]$.

[4] However, in our results we shall not address this fact in order to keep the presentation not too technical.

is called a *full marginal reduction* of f_i if f_i is partially differentiable with respect to its i-th variable and

$$D_i f_i(\mathbf{x}) = t_i(x_i, q(\underline{\mathbf{x}})) \text{ for every } \mathbf{x} \in \mathbf{X}. \tag{1}$$

Note that the existence of a full marginal reduction of f_i implies that f_i is continuous in each variable.

The most important property of the full marginal reductions is that in any equilibrium $\mathbf{e}$ for all $i \in N$

$$e_i \in \mathrm{Int}(X_i) \Rightarrow t_i(e_i, q(\mathbf{e})) = 0; \tag{2}$$

$$e_i = 0 \Rightarrow t_i(e_i, q(\mathbf{e})) \leq 0; \tag{3}$$

$$X_i = [0, m_i] \wedge e_i = m_i \Rightarrow t_i(e_i, q(\mathbf{e})) \geq 0. \tag{4}$$

For $i \in N$, a linear co-strategy function $q : \mathbf{X} \to \mathbb{R}$ and $Z \subseteq \mathbf{X}$ we define

$$\mathcal{W}_i(Z; q) \ := \ \{(x_i, q(\mathbf{x})) \mid \mathbf{x} \in Z\} \subseteq X_i \times Y_q.$$

Let Z be a subset of $\mathbf{X}$ and $i \in N$. Any two-tuple $(t_i; q)$ where q is a linear co-strategy function and $t_i : V_i \to \mathbb{R}$ a function with $\mathcal{W}_i(Z, q) \subseteq V_i \subseteq X_i \times Y_q$, is called a *marginal reduction* of f_i on Z (with domain V_i) if for every $\mathbf{x} \in Z$, f_i is partially differentiable w.r.t. its i-th variable at $\mathbf{x}$ and $D_i f_i(\mathbf{x}) = t_i(x_i, q(\mathbf{x}))$. So a full marginal reduction of f_i is nothing else than a marginal reduction of f_i on $\mathbf{X}$ with domain $X_i \times Y_q$.

3.2. Quasi-concave Conditional Payoff Functions

Proposition 1. *1. Sufficient for all conditional payoff functions of player i to be concave is that there exists a full marginal reduction $(t_i; q)$ of f_i with t_i decreasing in its first variable and in its second variable.*

2. Sufficient for all conditional payoff functions of player i to be strictly concave is that there exists a full marginal reduction $(t_i; q)$ of f_i with t_i decreasing in its first and second variable, and strictly in at least one of the variables. ⋄

Proof. We prove 1; the proof of 2 is analogous. Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. Write $a = \sum_l q_l z_l$. Concavity of $f_i^{(\mathbf{z})}$ is equivalent to decreasingness of $Df_i^{(\mathbf{z})}$. Take $x_i, x_i{}' \in X_i$ with $x_i < x_i{}'$. We have $Df_i^{(\mathbf{z})}(x_i') = t_i(x_i{}', q_i x_i{}' + a) \leq t_i(x_i, q_i x_i + a) = Df_i^{(\mathbf{z})}(x_i)$. □

Proposition 2. *Sufficient for all conditional payoff functions of player i to be strictly quasi-concave is that there exists a full marginal reduction $(t_i; q)$ of f_i such that for all $x_i \in X_i$ and $y \in Y_q$ with $x_i \leq y$*

- $D_1 t_i(x_i, y) < 0$;
- $t_i(x_i, y) = 0 \ \Rightarrow \ D_2 t_i(x_i, y) \leq 0$;
- $D_1 t_i, D_2 t_i : X_i \times Y \to \mathbb{R}$ *are continuous.* ⋄

Proof. Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. Write $a = \sum_l q_l z_l$. Consider $h = f_i^{(\mathbf{z})} \restriction \mathrm{Int}(X_i)$. We have $h'(x_i) = t_i(x_i, q_i x_i + a)$ and $h''(x_i) = D_1 t_i(x_i, q_i x_i + a) + q_i D_2 t_i(x_i, q_i x_i + a)$. So h is a two times continuously differentiable function on the open interval $\mathrm{Int}(X_i)$. For all x_i in this interval we have $h'(x_i) = 0 \ \Rightarrow \ h''(x_i) < 0$. Théorème 9.2.6. in Truchon(1987)) guarantees that h is strictly quasi-concave. As $f_i^{(\mathbf{z})}$ is continuous, it follows that also $f_i^{(\mathbf{z})}$ is strictly quasi-concave. □

In the next proposition we use the following principle for a function $g : \mathbb{R}_+ \to \mathbb{R}$: *sufficient for g to be quasi-concave is that there exists $s \geq 0$ such that g is decreasing on $[s, +\infty[$, concave on $[0, s[$ and continuous in s; also, sufficient for g to be strictly quasi-concave is that there exists $s \geq 0$ such that g is strictly decreasing on $[s, +\infty[$, strictly concave on $[0, s[$ and continuous in s.*

Proposition 3. *Suppose $X_i = \mathbb{R}_+$, f_i is continuous and let $w > 0$. Suppose there exists a marginal reduction $(t_i; \alpha)$ of f_i on $\{\mathbf{x} \in \mathbf{X} \mid \underline{\mathbf{x}} < w\}$ with domain $[0, w[\times [0, w[$. Fix $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ with $\underline{\mathbf{z}} < w$.*

1. *Sufficient for $f_i^{(\mathbf{z})}$ to be quasi-concave is that $f_i^{(\mathbf{z})}$ is decreasing on $[w - \underline{\mathbf{z}}, +\infty[$, that for all $y \in [0, w[$, the function $t_i(\cdot, y)$ is decreasing on $[0, y]$ and that for all $x_i \in [0, w[$ the function $t_i(x_i, \cdot)$ is decreasing on $[x_i, w[$.*
2. *Sufficient for $f_i^{(\mathbf{z})}$ to be strictly quasi-concave is that $f_i^{(\mathbf{z})}$ is strictly decreasing on $[w - \underline{\mathbf{z}}, +\infty[$ and that at least one of the following conditions holds*
   - *for all $y \in [0, w[$, the function $t_i(\cdot, y)$ is strictly decreasing on $[0, y]$ and for all $x_i \in [0, w[$, the function $t_i(x_i, \cdot)$ is decreasing on $[x_i, w[$;*
   - *for all $y \in [0, w[$, the function $t_i(\cdot, y)$ is decreasing on $[0, y]$ and for all $x_i \in [0, w[$, the function $t_i(x_i, \cdot)$ is strictly decreasing on $[x_i, w[$.* $\diamond$

Proof. We prove 1. The proof of 2 is analogous. Note that $D_i f_i^{(\mathbf{z})}(x_i) = t_i(x_i, x_i + \underline{\mathbf{z}})$ $(0 \leq x_i < w - \underline{\mathbf{z}})$. Let $\gamma := w - \underline{\mathbf{z}} > 0$. By the above principle it is sufficient to prove that $f_i^{(\mathbf{z})} \upharpoonright [0, \gamma[$ is concave. Let $a_i, b_i \in [0, \gamma[$ with $a_i < b_i$. Now $a_i + \underline{\mathbf{z}}, b_i + \underline{\mathbf{z}} \in [0, w[$. We obtain $D_i f_i^{(\mathbf{z})}(a_i) = t_i(a_i, a_i + \underline{\mathbf{z}}) \leq t_i(a_i, b_i + \underline{\mathbf{z}}) \leq t_i(b_i, b_i + \underline{\mathbf{z}}) = D f_i^{(\mathbf{z})}(b_i)$. $\square$

Proposition 3 may be useful for situations where conditional payoff functions $f_i^{(\mathbf{z})}$ are decreasing for x_i large enough.

3.3. Equilibrium Existence

In this subsection we consider a game in strategic form without the extra assumptions made about the strategy sets in Subsection 2.1.

The Nikaido-Isoda theorem states that sufficient conditions for the existence of an equilibrium are: *strategy sets are convex compact subsets of a finite dimensional normed real linear space, payoff functions are continuous and the set of maximisers of each conditional payoff function is convex.*

The next theorem concerns a simple generalisation of this result allowing for non-compact strategy sets. Its proof directly follows from the following fundamental observation for a game in strategic form Γ: *let, for each player i, W_i be a non-empty subset of X_i and let Γ' be the game in strategic form with the same player set, W_i as strategy set and $f_i \upharpoonright \mathbf{W}$ as payoff function for player i, then,*

$$E(\Gamma) \cap \mathbf{W} \subseteq E(\Gamma');$$

$$\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright W_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})} \ (i \in N, \mathbf{z} \in \mathbf{X}_{\hat{\imath}}) \ \Rightarrow \ E(\Gamma') \subseteq E(\Gamma). \tag{5}$$

Theorem 1. *Suppose for each player i there exists a non-empty subset W_i of X_i such that for every $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$*

$$\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright W_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})}.$$

Then the following conditions are sufficient for the existence of an equilibrium:

a. *every W_i is a convex compact subset of a finite dimensional normed real linear space;*
b. *every $f_i \restriction \mathbf{W}$ is continuous;*
c. *the set of maximisers of each $f_i^{(\mathbf{z})} \restriction W_i$ is convex.* $\diamond$

3.4. Equilibrium Semi-uniqueness

The following proposition is a simple improvement of the semi-uniqueness result in Corchón(2001). It is good to provide here again a proof (in three lines).

Proposition 4. *Sufficient for $\#E \leq 1$ to hold is that for every $i \in N$ there exists a full marginal reduction $(t_i; q)$ of f_i for which t_i is strictly decreasing in its first variable and decreasing in its second.* $\diamond$

Proof. By contradiction. So suppose $\mathbf{a}$ and $\mathbf{b}$ are two equilibria. We may suppose $q(\mathbf{b}) \geq q(\mathbf{a})$. Fix $j \in N$ with $b_j > a_j$. We obtain $t_j(a_j, q(\mathbf{a})) \geq t_j(a_j, q(\mathbf{b})) > t_j(b_j, q(\mathbf{b}))$. But, by (2) we obtain the contradiction $t_j(a_j, q(\mathbf{a})) \leq 0 \leq t_j(b_j, q(\mathbf{b}))$. $\square$

Here is an improvement of Proposition 4 (with exactly the same proof):

Proposition 5. *Suppose for every $i \in N$ there exists a marginal reduction $(t_i; q)$ of f_i on E with domain $\{(x_i, q(\mathbf{x})) \mid \mathbf{x} \in E\}$. Sufficient for $\#E \leq 1$ to hold is that for every $\mathbf{a}, \mathbf{b} \in E$ with $q(\mathbf{b}) \geq q(\mathbf{a})$ and $i \in N$ one has: $b_i > a_i \;\Rightarrow\; t_i(a_i, q(\mathbf{a})) > t_i(b_i, q(\mathbf{b}))$.* $\diamond$

Here is a variant of Proposition 4 (with again the same proof when q is replaced by α):

Proposition 6. *Suppose for every $i \in N$ there exists a full marginal reduction $(t_i; \alpha)$ of f_i. Sufficient for $\#E \leq 1$ to hold is that for every $i \in N$ and $x_i \in X_i$, the function $t_i(x_i, \cdot)$ is decreasing on $\{y \in Y \mid y \geq x_i\}$ and that for every $i \in N$ and $y \in X_i$, the function $t_i(\cdot, y)$ is decreasing on $\{x_i \in X_i \mid x_i \leq y\}$.* $\diamond$

For variants that can deal with payoff functions f_i that are left and right differentiable with respect to its i-th variable see von Mouche(2011).

3.5. Decreasing Best Reply Correspondences

Theorem 2. *Suppose $(t_i; q)$ is a full marginal reduction of f_i. Let $\mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ be the essential domain of R_i.*[5] *Sufficient for every single-valued selection of the correspondence*

$$R_i : \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})} \multimap X_i$$

to be decreasing,[6] *is that for all $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z} < \mathbf{z}'$, $x \in R_i(\mathbf{z}), x' \in R_i(\mathbf{z}')$ one has*

$$x < x' \;\Rightarrow\; t_i(x', q(x'; \mathbf{z}')) < t_i(x, q(x; \mathbf{z})).$$

In particular the following property is sufficient: t_i is decreasing in the first and in the second variable and strictly decreasing in at least one of them. $\diamond$

Proof. By contradiction. So suppose $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{\mathrm{ess}}$ with $\mathbf{z} < \mathbf{z}'$, $x \in R_i(\mathbf{z})$, $x' \in R_i(\mathbf{z}')$, $x < x'$. It follows that $Df_i^{(\mathbf{z})}(x) \leq 0$ and $Df_i^{(\mathbf{z}')}(x') \geq 0$. So we obtain the contradiction $t_i(x, q(x; \mathbf{z})) \leq 0 \leq t_i(x', q(x'; \mathbf{z}'))$. $\square$

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[5] I.e., the set $\{\mathbf{z} \in \mathbf{X}_{\hat{\imath}} \mid R_i(\mathbf{z}) \neq \emptyset\}$.

[6] I.e., for all $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z} < \mathbf{z}'$ (i.e. $z_l \leq z_l'$ for all $l \neq i$ with at least one strict inequality), $x \in R_i(\mathbf{z}), x' \in R_i(\mathbf{z}')$ one has $x \leq x'$.

4. Oligopolies

4.1. Market Satiation Points

If the price function has a market satiation point v, then we define

$$\mathcal{Y} := \begin{cases} [0, v] \text{ if } v < +\infty, \\ v \text{ if } v = +\infty. \end{cases}$$

Proposition 7. *If p is concave with a non-zero market satiation point v, then for each $\mathbf{e} \in E$ one has $\underline{\mathbf{e}} \leq v$.* $\diamond$

Proof. If $v = +\infty$, then the statement is evident. Also if $v = m$ in case $Y = [0, \sum_l m_l]$, the statement is evident. Now suppose $v \in \mathrm{Int}(Y)$. As p is concave, p is continuous in v and therefore $p(v) = 0$. As $v > 0$, there is $a \in]0, v[$ with $p(a) > 0$. As p is concave, it follows that $p(y) < 0$ $(y > v)$.

Now suppose $\mathbf{e} \in E$. We prove by absurd that $\underline{\mathbf{e}} \leq v$. So suppose $\underline{\mathbf{e}} > v$. Fix $j \in N$ such that $e_j > 0$. Now $f_j^{(\mathbf{e}_i)}(e^j) = p(\underline{\mathbf{e}})e_j - c_j(e_j) < -c_j(e_j) = f_j^{(\mathbf{e}_i(}(0)$, which is a contradiction. $\square$

The following result provides a weak condition that guarantees that each equilibrium $\mathbf{e}$ satisfies $\underline{\mathbf{e}} \in \mathcal{Y}$.

Proposition 8. *Suppose p has a finite market satiation point v and*

$$c_i(x_i) > c_i(0) \ (i \in N, \ x_i \in X_i \textit{ with } x_i \geq v/n).^7$$

Then for each equilibrium $\mathbf{e}$

1. $\mathbf{e} \neq \mathbf{0} \Rightarrow p(\underline{\mathbf{e}}) > 0$;
2. $\underline{\mathbf{e}} \leq v$;
3. $[v \neq +\infty, p(v) \leq 0] \ \Rightarrow \ \underline{\mathbf{e}} < v$. $\diamond$

Proof. 1. By way of contradiction assume that $\mathbf{e} \neq \mathbf{0}$ and $p(\mathbf{e}) \leq 0$. Then

$$f_i(\mathbf{e}) = p(\underline{\mathbf{e}})e_i - c_i(e_i) \leq -c_i(e_i) \ (i \in N).$$

As $\mathbf{e} \in E$, it follows for every $i \in N$ that $f_i(\mathbf{e}) \geq f_i(0; \mathbf{e}_{\hat{i}}) = -c_i(0)$ and therefore $c_i(e_i) \leq c_i(0)$. If i has a capacity constraint and $m_i < v/n$, then $e_i \leq m_i < v/n$. If i has a capacity constraint and $m_i \geq v/n$, then $e_i \geq v/n$ would imply the contradiction $c_i(e_i) > c_i(0)$. If i does not have a capacity constraint, then $e_i \geq v/n$ again would imply the contradiction $c_i(e_i) > c_i(0)$. So $e_i < v/n$ $(i \in N)$. But now $0 < \underline{\mathbf{e}} < \sum_{l=1}^n v/n = v$ and therefore $p(\underline{\mathbf{e}}) > 0$, which is a contradiction.

2. This follows from 1.

3. By 2 we have $\underline{\mathbf{e}} \leq v$. If $\underline{\mathbf{e}} = v$ would hold, then $\mathbf{e} \neq \mathbf{0}$ and $p(\underline{\mathbf{e}}) = p(v) \leq 0$, which is a contradiction with 1. $\square$

Almost all results for oligopolies in the literature deal with decreasing price functions, and therefore with price functions that have a market satiation point. Results like Proposition 8 (assuming a weak monotonicity assumption for the cost functions) imply that in many results only the properties of p on $\mathcal{Y}$ and of c_i on $\mathcal{Y} \cap X_i$ matter; in such cases one may call $\mathcal{Y}$ (resp. $\mathcal{Y} \cap X_i$) also the *relevant domain*

[7] This implies that $v \neq 0$.

of p (resp. c_i). This implies that various results in the literature, like Theorem 7 below, can be improved by taking these domains into consideration. In order to make this point clearer, consider for example the following result for a duopoly without capacity constraints: if $p(y) = 7 - y$ and cost functions are convex, strictly increasing and continuously differentiable, then there exists at most one equilibrium. This result in fact follows from Theorem 7 below. As this theorem does not deal with $\mathcal{Y}$ explicitly, it does not imply the following result: if $p(y) = 7 - y$ ($0 \leq y \leq 7$), $p(y) \leq 0$ ($y \geq 7$) and cost functions are convex, strictly increasing and continuously differentiable, then there exists at most one equilibrium. But this result follows from Theorem 7 together with the following result:

Theorem 3. *Consider an oligopoly where the price function p has a (possibly infinite) market satiation point and each cost function is increasing and has 0 as unique minimiser. Let*

$$\tilde{p} \;:=\; \max(p, 0).$$

Then if we replace in the oligopoly the price function p by $\tilde{p}$, the set of equilibria does not change. $\diamond$

Proof. Let Γ be the original game and Γ' the modified game. Let v be the market satiation point of p. Note that $\tilde{p}$ also has a market satiation point and that this again is v. Proposition 8 guarantees

$$\mathbf{e} \in E(\Gamma) \setminus \{\mathbf{0}\} \;\Rightarrow\; [p(\mathbf{e}) > 0,\ \underline{\mathbf{e}} \leq v],$$

$$\mathbf{e} \in E(\Gamma') \setminus \{\mathbf{0}\} \;\Rightarrow\; [\tilde{p}(\mathbf{e}) > 0,\ \underline{\mathbf{e}} \leq v].$$

Fix $i \in N$, $x_i \in X_i$. Denote the profit function of firm i in the modified game by $\tilde{f}_i$.

- $E(\Gamma) \subseteq E(\Gamma')$: suppose $\mathbf{e} \in E$. Let $a = \sum_{l=1, l \neq i}^{n} e_l$. If $p(x_i + a) > 0$, then, $\tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(x_i + a)x_i - c_i(x_i) = p(x_i + a)x_i - c_i(x_i) = f_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq f_i(e_i; \mathbf{e}_{\hat{\imath}}) = p(e_i + a)e_i - c_i(e_i) = \tilde{p}(e_i + a)e_i - c_i(e_i) = \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}})$. If $p(x_i + a) \leq 0$, then noting that $x_i + a = 0$ or $x_i \geq e_i$, we obtain $\tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(x_i + a)x_i - c_i(x_i) \leq -c_i(x_i) \leq -c_i(e_i) \leq \tilde{p}(e_i + a)e_i - c_i(e_i) = \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}})$.
- $E(\Gamma') \subseteq E(\Gamma)$: suppose $\mathbf{e} \in E(\Gamma')$. Let $a = \sum_{l=1, l \neq i}^{n} e_l$. Note that $0 < \tilde{p}(e_i + a) = p(e_i + a)$. If $p(x_i + a) > 0$, then $f_i(x_i; \mathbf{e}_{\hat{\imath}}) = p(x_i + a)x_i - c_i(x_i) = \tilde{p}(x_i + a)x_i - c_i(x_i) = \tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(e_i + a)e_i - c_i(e_i) = p(e_i + a)e_i - c_i(e_i) = f_i(e_i; \mathbf{e}_{\hat{\imath}})$. If $p(x_i + a) \leq 0$, then $f_i(x_i; \mathbf{e}_{\hat{\imath}}) = p(x_i + a)x_i - c_i(x_i) \leq \tilde{p}(x_i + a)x_i - c_i(x_i) = \tilde{f}_i(x_i; \mathbf{e}_{\hat{\imath}}) \leq \tilde{f}_i(e_i; \mathbf{e}_{\hat{\imath}}) = \tilde{p}(e_i + a)e_i - c_i(e_i) = p(e_i + a)e_i - c_i(e_i) = f_i(e_i; \mathbf{e}_{\hat{\imath}})$. $\square$

A problem is that dealing with $p \restriction \mathcal{Y}$ and $c_i \restriction X_i \cap \mathcal{Y}$ complicates the proofs (and the presentation). What we would like to have are general results that enable to obtain us to derive simply from a result in terms of p and the cost functions c_i a variant in terms of $p \restriction \mathcal{Y}$ and $c_i \restriction X_i \cap \mathcal{Y}$. However, Theorem 3 is a first step into this direction. Its usefullness will be illustrated by deriving an improvement of Theorem 7.

Finally, we provide here a simple result in case p has 0 as market satiation point.

Proposition 9. *Suppose 0 is the market satiation point of p and each cost function has 0 as a minimiser. Then:*

1. $\mathbf{0}$ *is an equilibrium.*
2. *If* 0 *even is the unique minimiser of each cost function, then* $\mathbf{0}$ *is the unique equilibrium.* $\diamond$

Proof. Note that $p(x_i)x_i \leq 0$ $(x_i \in X_i)$ and $c_i(0) \leq c_i(x_i)$ $(x_i \in X_i)$.

1. For every $x_i \in X_i$ we have $f_i^{(\mathbf{0})}(0) = -c_i(0) \geq -c_i(x_i) \geq p(x_i)x_i - c_i(x_i) = f_i^{(\mathbf{0})}(x_i)$. Thus $\mathbf{0} \in E$.

2. Having 1, we need to prove that $\mathbf{e} \in E \ \Rightarrow \ \mathbf{e} = 0$. So suppose $\mathbf{e} \in E$. Let $i \in N$. We shall prove by contradiction that $e_i = 0$. So suppose $e_i > 0$. As $\mathbf{e} \in E$, we have $\pi_i^{(\mathbf{e}^{\hat{\imath}})}(e_i) \geq f_i^{(\mathbf{e}^{\hat{\imath}})}(0)$, i.e. $p(\underline{\mathbf{e}})e_i - c_i(e_i) \geq -c_i(0)$. So $p(\underline{\mathbf{e}})e_i \geq c_i(e_i) - c_i(0)$. In the last inequality the left-hand side is non-positive and the right-hand side is positive, which is absurd. $\square$

4.2. Fisher-Hahn and Related Conditions

The following conditions (and its variants) play an important role in uniqueness results for oligopolies.

– p is differentiable, c_i is twice differentiable and

$$Dp(y) - D^2c_i(x_i) < 0 \ (x_i \in X_i, \ y \in Y); \tag{6}$$

$$Dp(y) - D^2c_i(x_i) < 0 \ (x_i \in X_i, \ y \in Y, \ x_i \leq y). \tag{7}$$

– p is twice differentiable and

$$Dp(y) + x_iD^2p(y) \leq 0 \ (x_i \in X_i, \ y \in Y); \tag{8}$$

$$Dp(y) + x_iD^2p(y) < 0 \ (x_i \in X_i, \ y \in Y); \tag{9}$$

$$Dp(y) + x_iD^2p(y) \leq 0 \ (x_i \in X_i, \ y \in Y, \ x_i \leq y). \tag{10}$$

$$Dp(y) + yD^2p(y) \leq 0 \ (y \in Y). \tag{11}$$

– p is twice differentiable and

$$2Dp(y) + yD^2p(y) \leq \ 0 \ (y \in Y); \tag{12}$$

$$2Dp(y) + yD^2p(y) < \ 0 \ (y \in Y); \tag{13}$$

$$p(y)D^2p(y) - (Dp(y))^2 \leq 0 \ (y \in Y). \tag{14}$$

If p and c_i are differentiable, then we define $(t_i; \alpha)$ with $t_i : X_i \times Y \to \mathbb{R}$ by

$$t_i(x_i, y) \ := \ Dp(y)x_i + p(y) - Dc_i(x_i). \tag{15}$$

This $(t_i; \alpha)$ is a full marginal reduction of f_i. So if p is differentiable and c_i is twice differentiable, then (15) implies for all $x_i \in X_i$ and $y \in Y$

$$Dp(y) - D^2c_i(x_i) = D_1t_i(x_i, y).$$

Besides, if p is twice differentiable and c_i is differentiable, then

$$Dp(y) + D^2p(y)x_i = D_2t_i(x_i, y).$$

Also note that for a twice differentiable price function: (12) is equivalent to concavity of the aggregate revenue r; (13) is equivalent to strict concavity of r; (14) is for positive p equivalent to log-concavity of p.

We also refer to condition (6) as the *first Fisher-Hahn condition*, to (9) as the *second Fisher-Hahn condition*, to (8) as the *weak second Fisher-Hahn condition.* Condition (11) is called the *marginal revenue condition* and plays, as first shown in Novshek(1985), an important role in equilibrium existence proofs. The first Fisher-Hahn condition is used in various results which allow for non-convex cost functions. In these results non-convexity of the cost functions is compensated by monotonicity properties of the price function. Proposition 10(6) below implies that these results deal with situations where each firm has a capacity constraint or where the market satiation point exists and is finite. Proposition 10(1) shows that in case $Y = \mathbb{R}_+$ the weak second Fisher-Hahn condition implies the marginal revenue condition. Proposition 10(1,4) shows that the marginal revenue condition implies that r is concave. Of course, we have the implications (6) $\Rightarrow$ (7) and (8) $\Rightarrow$ (10). Here are some other relations:

Proposition 10. *1. (11) implies (10). In case $Y = \mathbb{R}_+$, (10) and (11) are equivalent.*

2. (10) implies $Dp \leq 0$. Also (11) implies $Dp \leq 0$.
3. (12) implies $Dp(y) \neq 0$ $(y \neq 0)$. And (13) implies $Dp(y) < 0$ $(y \neq 0)$.
4. (10) implies (12).
5. Sufficient for (8) to hold is that p is twice differentiable, decreasing and concave.
6. Suppose $y = \mathbb{R}_+$. If $p > 0$ is decreasing, then (6) implies that cost functions are convex.
7. If $Y = \mathbb{R}_+$ and p is not constant, then (14) and (12) cannot hold together. $\diamond$

Proof. 1. First statement: suppose (11) holds and let $x_i \in X_i$ and $y \in Y$ with $x_i \leq y$. So $Dp(y) + yD^2p(y) \leq 0$. By 1, $Dp \leq 0$. If, $D^2p(y) \leq 0$, then $Dp(y) + x_iD^2p(y) \leq 0$. If $D^2p(y) > 0$, then $Dp(y) + x_iD^2p(y) \leq -yD^2p(y) + x_iD^2p(y) = (x_i - y)D^2p(y) \leq 0$. The proof of the second statement now follows immediately.

2. The first statement follows by taking $x_i = 0$. So, by 1, also the second statement holds.

3. We prove the first statement; the proof of the second is analogous. Fix $y \neq 0$. As r is concave and differentiable at y, it follows that $r(0) \leq r(y) + Dr(y)(0 - y)$, i.e. $0 \leq p(y)y - y(Dp(y)y + p(y))$ and therefore $y^2Dp(y) \leq 0$. As $y \neq 0$, it follows that $Dp(y) \leq 0$.

4. By 2, $Dp \leq 0$, so (12) holds.

5. Evident.

6. As p is decreasing, $Dp \leq 0$ holds. Let $\epsilon > 0$. As $\lim_{y \to +\infty} \frac{p(y)-p(7)}{y-7} = 0$, there exists $y > 7$ such that $\frac{p(y)-p(7)}{y-7} \geq -\epsilon$. The first mean value theorem implies the existence of $\xi \in]7, y[$ with $Dp(\xi) = \frac{p(y)-p(7)}{y-7}$. Thus $Dp(\xi) \geq -\epsilon$. It follows that $\sup_{y \in \mathbb{R}_{++}} Dp(y) = 0$. By (6) $Dp(y) < D^2c_i(x_i)$ $(x_i \in X_i,\ y \in Y)$. It follows that $D^2c_i(x_i) \geq 0$ $(x_i \in X_i)$. So c_i is convex.

7. By way of contradiction suppose (14) and (12) hold. So p is log-concave and r is concave. As r is concave, $r(0) = 0 < r(71)$ and $r \geq 0$ it follows that $\lim_{y \to +\infty} r(y)$ exists as element of $\mathbb{R} \cup \{+\infty\}$ and $\lim_{y \to +\infty} r(y) > 0$. By 2, p is decreasing. This in turn implies that there exists $z > 0$ with $Dp(z) < 0$. Let $\alpha := \ln p(z)$ and

$\beta := D \ln p(z) = \frac{Dp(z)}{p(z)} < 0$. Let $l : \mathbb{R} \to \mathbb{R}$ be defined by $l(y) := \alpha + \beta(y - z)$. Since $l(z) = \ln p(z)$ and, on $\mathbb{R}_{++}$, l is affine and $\ln p$ is concave, it follows that $\ln p(y) \leq l(y)$ ($y \in \mathbb{R}_{++}$). Therefore, $r(y) = p(y)y \leq e^{\alpha - \beta z + \beta y} y$ ($y \in \mathbb{R}_{++}$). So we obtain a contradiction because $\lim_{y \to +\infty} r(y) = 0$. □

Condition (12) does not imply (10) and therefore by Proposition 10(1) nor the marginal revenue condition: for $p(y) = 1/(y+1)$, r is concave and so (12) holds and one has $Dp(y) + x_i D^2 p(y) = (y+1)^{-3}(2x_i - y - 1)$.

4.3. Quasi-concave Conditional Profit Functions

The proof of the following result is a good exercise in convex analysis and should be well-known in oligopoly theory:

Proposition 11. *For $a \in X_i$, let $r_a : (X_i - \{a\}) \cap X_i \to \mathbb{R}$ be defined by $r_a(x_i) := p(a + x_i)x_i$. So $r_0 = r$.*

1. *Sufficient for r to be (strictly) concave is that p is concave and (strictly) decreasing.*
2. *Sufficient for each r_a to be (strictly) concave is that r is (strictly) concave.*
3. *Suppose r is concave and c_i is convex. Then each conditional profit function of firm i is concave. If, in addition, r is strictly concave or c_i is strictly convex, then each conditional profit function of firm i is (strictly) concave.* ⋄

Proposition 1 implies the following.

Corollary 1. *If the first and Fisher-Hahn condition and weak second Fisher-Hahn condition hold, then each conditional profit function of firm i is strictly concave.* ⋄

Proposition 12. *Fix $i \in N$. Suppose p and c_i are twice continuously differentiable, p is decreasing and log-concave and c_i is increasing.*

1. *For every $x_i \in X_i$ and $y \in Y$: $t_i(x_i, y) = 0 \Rightarrow D_2 t_i(x_i, y) \leq 0$.*
2. *If the first Fisher-Hahn condition holds, then each conditional profit function of firm i is strictly quasi-concave.* ⋄

Proof. 1. As c_i is increasing, we have $Dc_i(x_i) \geq 0$ and as p is decreasing, we have $Dp(y) \leq 0$. Also $t_i(x_i, y) = 0 = x_i Dp(y) + p(y) - Dc_i(x_i)$. As p is log-concave, (14) holds and we obtain $D_2 t_i(x_i, y) = Dp(y) + x_i D^2 p(y) \leq Dp(y) + x_i \frac{(Dp(y))^2}{p(y)} = \frac{Dp(y)}{p(y)}(p(y) + x_i Dp(y)) = \frac{Dp(y)}{p(y)} Dc_i(x_i) \leq 0$.

2. This follows from Proposition 2. That its conditions hold, follows from 1. □

Proposition 13. *Suppose p has a finite market satiation point v and $X_i = \mathbb{R}_+$. Further suppose*

a. *p is continuous in v and $p(y) = 0$ ($y \geq v$);*
b. *c_i is increasing and continuous at each point of $[0, v]$;*
c. *r is concave on $[0, v]$ and c_i is convex on $[0, v]$.*

Then each conditional profit function is quasi-concave. ⋄

Proof. If $\underline{\mathbf{z}} \geq v$, then $f_i^{(\mathbf{z})} = -c_i$ is quasi-concave. Now suppose $\underline{\mathbf{z}} < v$. We use the principle mentioned before Proposition 3. On $[v - \underline{\mathbf{z}}, +\infty[$ the function $f_i^{(\mathbf{z})}$ equals $-c_i$ and therefore is decreasing there. Now consider $f_i^{(\mathbf{z})}$ on $[0, v - \underline{\mathbf{z}}[$. For $x_i \in [0, v - \underline{\mathbf{z}}[$ one has $f_i^{(\mathbf{z})}(x_i) = p(x_i + \underline{\mathbf{z}})x_i - c_i(x_i)$. As $f_i^{(\mathbf{z})}$ is continuous in $v - \underline{\mathbf{z}}$ and c_i is convex on $[0, v - \underline{\mathbf{z}}[$, the proof is complete if $x_i \mapsto p(x_i + \underline{\mathbf{z}})x_i$ is concave on $[0, v - \underline{\mathbf{z}}[$. Well, this follows from (a variant of) Proposition 11(2). □

Theorem 4. *Suppose p has a non-zero finite market satiation point v and $X_i = \mathbb{R}_+$. Further suppose*

a. p is continuous, $p(y) = 0$ $(y \geq v)$ and p is on $[0, v[$ twice differentiable;
b. c_i is increasing and on $[0, v[$ twice differentiable;
c. for every $x_i, y \in [0, v[$ with $x_i \leq y$

$$Dp(y) - D^2 c_i(x_i) < 0, \ \ yD^2 p(y) + Dp(y) \leq 0.$$

Then p is decreasing and each conditional profit function is quasi-concave. ⋄

Proof. Let $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$. If $\underline{\mathbf{z}} \geq v$, then $f_i^{(\mathbf{z})} = -c_i$. As c_i is increasing, $f_i^{(\mathbf{z})}$ is quasi-concave in this case. Now suppose $\underline{\mathbf{z}} < v$. Define $t_i : X_i \times [0, v[$ by $t_i(x_i, y) = Dp(y)x_i + p(y) - Dc_i(x_i)$. Fix $x_i, y \in [0, v[$. As in the proof of Proposition 10(1) we see that c implies: for every $x_i, y \in [0, v[$ with $x_i \leq y$ it holds that $x_i D^2 p(y) + Dp(y) \leq 0$. In particular, $Dp(y) \leq 0$ $(0 \leq y < v)$. It follows that p is decreasing. As $Dp(y) - D^2 c_i(a) < 0$ $(0 \leq a \leq y)$, it follows that $t_i(\cdot, y)$ is strictly decreasing on $[0, y]$. As $D^2 p(a)x_i + Dp(a) \leq 0$ $(x_i \leq a \leq w)$, it follows that $t_i(x_i, \cdot)$ is decreasing on $[x_i, w[$. Proposition 3(1) applies and guarantees that $f_i^{(\mathbf{z})}$ is quasi-concave. □

4.4. Equilibrium Existence

Theorem 1 is central for the following fundamental existence result.

Theorem 5. *Suppose all profit functions are continuous and all conditional profit functions are quasi-concave. Then the additional following condition is sufficient for the existence of an equilibrium: if there is a firm i without capacity constraint, then the price function p is decreasing and for each such firm i there exists $\overline{x}_i > 0$ with $r(x) \leq c_i(x) - c_i(0)$ $(x \geq \overline{x}_i)$.* ⋄

Proof. If firm i has a capacity constraint, then let $W_i = X_i$ and if firm i does not, then let $W_i = [0, \overline{x}_i]$. Now for each $\mathbf{z} \in \mathbf{X}_{\hat{\imath}}$ we prove the inclusion

$$\operatorname{argmax} f_i^{(\mathbf{z})} \upharpoonright W_i \subseteq \operatorname{argmax} f_i^{(\mathbf{z})}.$$

Well, this is trivial if i has a capacity constraint. Now suppose i does not have. With $r_{\underline{\mathbf{z}}} : \mathbb{R}_+ \to \mathbb{R}$ defined by $r_{\underline{\mathbf{z}}}(x_i) = p(x_i + \underline{\mathbf{z}})x_i$, we have to prove

$$\operatorname{argmax} (r_{\underline{\mathbf{z}}} - c_i) \upharpoonright W_i \subseteq \operatorname{argmax} (r_{\underline{\mathbf{z}}} - c_i).$$

Suppose $x_i^\star$ is a maximiser of $(r_{\underline{\mathbf{z}}} - c_i) \upharpoonright W_i$ and let $x_i \in X_i$. If $x_i \in W_i$, then $(r_{\underline{\mathbf{z}}} - c_i)(x_i) \leq (r_{\underline{\mathbf{z}}} - c_i)(x_i^\star)$. And if $x_i \notin W_i$, then $x_i > \overline{x}_i$ and, using the decreasingness of p,

$$(r_{\underline{\mathbf{z}}} - c_i)(x_i) = (r - c_i)(x_i) + (p(x_i + \underline{\mathbf{z}}) - p(x_i))x_i \leq -c_i(0) + 0 = -c_i(0)$$

$$= (r_{\underline{\mathbf{z}}} - c_i)(0) \leq (r_{\underline{\mathbf{z}}} - c_i)(x_i^\star).$$

Denote the game by Γ and let Γ' be the game in strategic form with the player set N, with W_i as strategy sets and the $f_i \upharpoonright \mathbf{W}$ as payoff functions. (5) implies that $E(\Gamma') \subseteq E(\Gamma)$. Each W_i is non-empty, convex and compact. Also the set of maximisers of each $f_i^{(\mathbf{z})} \upharpoonright W_i$ is convex (as the set of maximisers $f_i^{(\mathbf{z})}$ is convex) and each $f_i \upharpoonright \mathbf{W}$ is continuous. Theorem 1 guarantees that $E(\Gamma') \neq \emptyset$ and so the proof is complete. □

4.5. Equilibrium Semi-uniqueness

Proposition 4 implies the following.

Corollary 2. *If for each firm i the first and weak second Fisher-Hahn conditions hold, then there exists at most one equilibrium.* ⋄

Concerning the signs of the inequalities in the Fisher-Hahn conditions, we note that $Dp(y) - D^2 c_i(x_i) \leq 0$ $(i \in N, x_i \in X_i, y \in Y)$ and $D^2 p(y) y + Dp(y) < 0$ $(y \in Y)$ are not sufficient for equilibrium semi-uniqueness. That this is true can be seen from the following example: $n = 2$, $X_1 = X_2 = \mathbb{R}_+$, $p(y) = -6y$ and $c_i(x_i) = -3x_i^2$.

In Vives(2001) a semi-uniqueness result is presented for oligopolies with log-concave price functions assuming the first Fisher-Hahn condition. This result addresses the relevant domain of p. Such a result does not follow from (a market satiation point variant of) Corollary 2 as for log-concave price functions the weak second Fisher-Hahn condition may not hold. The result in Vives(2001) was derived by an analysis of the backward best reply correspondences. In our next result we provide an improvement of this result.

Theorem 6. *Assume that price function p has a non-zero (may be infinite) market satiation point v, that p is continuous, decreasing and that p is log concave on* $\mathrm{Y} \setminus \{v\}$. *Also assume that each cost function is differentiable and strictly increasing. Finally assume that for all i and $y \in \mathrm{Y} \setminus \{v\}$ the function $X_i \to \mathbb{R}$ defined by*

$$x_i \mapsto Dp(y)x_i - Dc_i(x_i) \text{ is strictly decreasing.} \tag{16}$$

Then there exists at most one equilibrium. ⋄

Proof. First note that by Proposition 8, $\underline{\mathbf{e}} < v$ $(e \in E)$. Next note that $(t_i; \alpha)$ with $t_i : \{(x_i, \underline{\mathbf{x}}) \mid \mathbf{x} \in E\} \to \mathbb{R}$ defined by $t_i(x_i, \underline{\mathbf{x}})) = Dp(\underline{\mathbf{x}}))x_i + p(\underline{\mathbf{x}}) - Dc_i(x_i)$ is a marginal reduction of f_i with domain $\{(x_i, \underline{\mathbf{x}}) \mid \mathbf{x} \in E\}$. We now finish the proof by verifying the condition in Proposition 5.

So suppose $\mathbf{a}, \mathbf{b} \in \mathbf{E}$ with $\underline{\mathbf{b}} \geq \underline{\mathbf{a}}$ and $b_i > a_i$. We have to prove that

$$Dp(y_a)a_i + p(y_a) - Dc_i(a_i) > Dp(y_b)b_i + p(y_b) - Dc_i(b_i). \tag{17}$$

By (16) we have $Dp(y_a)a_i - Dc_i(a_i) > Dp(y_a)b_i - Dc_i(b_i)$. So also

$$Dp(y_a)a_i + p(y_a) - Dc_i(a_i) > Dp(y_a)b_i + p(y_a) - Dc_i(b_i).$$

We now prove that

$$Dp(y_a)b_i + p(y_a) \geq Dp(y_b)b_i + p(y_b). \tag{18}$$

Having this, (17) follows. Well, if $y_a = y_b$, then (18) holds. Now suppose $y_a < y_b$. As p is log-concave, $\frac{Dp}{p}$ is decreasing as well and therefore the function $\frac{Dp}{p} b_i + 1$ this is too. As $\mathbf{b}$ is an equilibrium and $b_i > 0$, we have $Dp(y_b)b_i + p(y_b) - Dc_i(b_i) \leq 0$. From this fact and from the increasingness of c_i we obtain $\frac{Dp(y_b)}{p(y_b)} b_i + 1 \geq \frac{Dc_i(b_i)}{p(y_b)} \geq 0$. As $p(y_a) \geq p(y_b) > 0$ we obtain

$$Dp(y_a)b_i + p(y_a) = p(y_a)(\frac{Dp(y_a)}{p(y_a)} b_i + 1) \geq p(y_a)(\frac{Dp(y_b)}{p(y_b)} b_i + 1)$$

$$\geq p(y_b)(\frac{Dp(y_b)}{p(y_b)} b_i + 1) = Dp(y_b)b_i + p(y_b). \;\square$$

4.6. Decreasing Best Reply Correspondences

Proposition 14. *Fix a firm i. Let $\mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ be the essential domain of R_i. Each of the following conditions separately is sufficient for every single-valued selection of the correspondence*

$$R_i : \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})} \multimap X_i$$

to be decreasing:

1. *p is differentiable and for every $x_i \in X_i$ the function $Y \to \mathbb{R}$ defined by $y \mapsto Dp(y)x_i + p(y)$ is strictly decreasing;*
2. *p is log-concave, and c_i is strictly increasing.* $\diamond$

Proof. 1. By contradiction. So assume $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z}_1 < \mathbf{z}_2$ and $x_k \in R_i(\mathbf{z}_k)$ $(k = 1, 2)$ with $x_1 < x_2$. Write $a_k = \underline{\mathbf{z_k}}$ $(k = 1, 2)$. Define the functions $w_1, w_2 : [x_1, x_2] \to \mathbb{R}$ by

$$w_k(\xi) \;:=\; p(\xi + a_k)\xi.$$

As $Dw_k(\xi) = Dp(\xi + a_k)\xi + p(\xi + a_k)$, we have $Dw_1 > Dw_2$. So $D(w_1 - w_2) > 0$. This implies that $w_1 - w_2$ is strictly increasing and therefore that $(w_1 - w_2)(x_2) > (w_1 - w_2)(x_1)$, i.e. that

$$p(x_2 + a_1)x_2 - p(x_2 + a_2)x_2 > p(x_1 + a_1)x_1 - p(x_1 + a_2)x_1. \tag{19}$$

As $x_k \in R_i(\mathbf{z}_k)$, we have

$$p(x_2 + a_1)x_2 - c_i(x_2) \leq p(x_1 + a_1)x_1 - c_i(x_1),$$

$$p(x_1 + a_2)x_1 - c_i(x_1) \leq p(x_2 + a_2)x_2 - c_i(x_2).$$

This implies

$$p(x_1 + a_2)x_1 - p(x_2 + a_2)x_2 \leq c_i(x_1) - c_i(x_2) \leq p(x_1 + a_1)x_1 - p(x_2 + a_1)x_2.$$

Therefore $p(x_1 + a_2)x_1 - p(x_2 + a_2)x_2 \leq p(x_1 + a_1)x_1 - p(x_2 + a_1)x_2$, which is a contradiction with (19).

2. By contradiction. So assume $\mathbf{z}, \mathbf{z}' \in \mathbf{X}_{\hat{\imath}}^{(\mathrm{ess})}$ with $\mathbf{z} < \mathbf{z}'$, $x \in R_i(\mathbf{z}), x' \in R_i(\mathbf{z}')$ and $x < x'$. Write $a = \underline{\mathbf{z}}, a' = \underline{\mathbf{z}}'$. As $x_i \in R_i(\mathbf{z})$, we have

$$p(x' + a')x' - c_i(x') \geq p(x + a')x - c_i(x). \tag{20}$$

As c_i is strictly increasing this leads to $p(x'+a')x' - p(x+a')x \geq c_i(x') - c_i(x) > 0$. Therefore

$$p(x'+a')x' > p(x+a')x. \tag{21}$$

As $\ln p$ is concave,

$$\ln p(x+a') - \ln p(x'+a') \geq \ln p(x+a) - \ln p(x'+a).$$

Indeed, using the three-cord-lemma

$$\ln p(x+a') - \ln p(x'+a') = (x-x')\frac{\ln p(x+a') - \ln p(x'+a')}{x+a'-(x'+a')}$$

$$\geq (x-x')\frac{\ln p(x+a) - \ln p(x'+a')}{x+a-(x'+a')} \geq (x-x')\frac{\ln p(x+a) - \ln p(x'+a)}{x+a-(x'+a)}$$

$$= (x-x')\frac{\ln p(x+a) - \ln p(x'+a)}{x-x'} = \ln p(x+a) - \ln p(x'+a).$$

This implies

$$\frac{p(x'+a)}{p(x'+a')} \geq \frac{p(x+a)}{p(x+a')}.$$

Multiplying the first term in the left hand side of (20) by $\frac{p(x'+a)}{p(x'+a')}$ and the first term in its right hand side by $\frac{p(x+a)}{p(x+a')}$, and noting that (21) and $c_i(x') > c_i(x)$ hold, we get

$$p(x'+a)x' - c_i(x') > p(x+a)x - c_i(x).$$

But this implies the contradiction $x \notin R_i(\mathbf{z})$). □

5. Three Powerfull Equilibrium Semi-uniqueness Results

5.1. Equilibrium Semi-uniqueness Result of Murphy-Sherali-Soyster

Theorem 7. *Consider a homogeneous Cournot oligopoly. Suppose no firm has a capacity constraint, the aggregate revenue function is concave and cost functions are convex. Also suppose the price function is continuously differentiable and strictly decreasing and cost functions are continuously differentiable. Each of the following conditions is sufficient for the existence of at most one Cournot equilibrium: (I) The aggregate revenue function is strictly concave. (II) All cost functions are strictly convex.* ⋄

Proposition 11(3) guarantees that in Theorem 7 all conditional profit functions are strictly concave. Therefore, we see with Theorem 1 that the following additional assumption is sufficient for equilibrium existence in Theorem 7: there exists $\overline{x} > 0$ such that $p(x)x \leq c_i(x) - c_i(0)$ for all $i \in N$ and $x \geq \overline{x}$.

The proof in Murphy et al.(1982)Murphy, Sherali, and Soyster is complex. In its proof some not so elementary results from mathematical programming were used. The difficulties in obtaining Theorem 7 are related to the fact that for the oligopolies therein (in case of twice differentiable price functions) the marginal revenue condition may not hold, and related with this that best reply correspondences may have non-decreasing single-valued selections. For example, consider the duopoly without capacity constraints with cost functions $c_1 = c_2 = x/100$ and price function

$p(y) = 1/(y+1)$. This duopoly satisfies the conditions of Theorem 1(I). It can be easily verified, e.g., that each best reply is 16 when the opponent produces 3 and it rises up to 21 when the opponent produces 8.

Proposition 10(7) show that p in Theorem 7 can not be log-concave.

Note that in Theorem 7 the price function p has a, may be infinite, market satiation point v and that in case $v \in]0,+\infty[$ one has $p(v) = 0$. The following result is an improvement of Theorem 7 in case of strictly increasing cost functions:

Theorem 8. *Suppose no firm has a capacity constraint, p has a market satiation point v with $p(v) = 0$ if $v \in]0,+\infty[$, the aggregate revenue function is concave on $\mathcal{Y}$ and cost functions are convex and strictly increasing. Also suppose the price function is continuously differentiable on $\mathcal{Y}$, strictly decreasing on $\mathcal{Y}$ and cost functions are continuously differentiable. Each of the following conditions is sufficient for the existence of at most one equilibrium: (I) The aggregate revenue function is strictly concave on $\mathcal{Y}$. (II) All cost functions are strictly convex.* $\diamond$

Proof. Denote the game by Γ. Note that $Y = \mathbb{R}_+$. If $v = +\infty$, then $\mathcal{Y} = Y$ and the result holds by Theorem 7. Now suppose $v < +\infty$. If $v = 0$, then the result holds by Proposition 9. Now suppose $v \in]0,+\infty[$. We have $\mathcal{Y} = [0,v]$ and $p(y) \leq 0$ $(y \geq v)$. As r is concave on $[0,v]$, $r(0) = r(v) = 0$ and $r > 0$ on $]0,v[$ it follows that $D^-r(v) < 0$ and, as $D^-r(v) = D^-p(v)v + p(v) = D^-p(v)$, also that $D^-p(v) < 0$. Let $\breve{p} : Y \to \mathbb{R}$ be defined by

$$\breve{p}(y) = \begin{cases} p(y) \text{ if } y \in [0,v], \\ D^-p(v)(y-v) - 107(y-v)^2 \text{ if } y \geq v. \end{cases}$$

Then $\breve{p}$ is continuously differentiable and strictly decreasing. The with $\breve{p}$ associated revenue function $\breve{r}$ is concave, and strictly concave if r is strictly concave on $\mathcal{Y}$. Denote the game where p is replaced by $\breve{p}$ by $\breve{\Gamma}$. $\breve{\Gamma}$ satisfies the general conditions in Theorem 7 and also I or II holds there. Therefore $\#E(\breve{\Gamma}) \leq 1$. Let $\tilde{\breve{\Gamma}}$ be the game obtained by replacing in $\breve{\Gamma}$ the price function $\breve{p}$ by $\max(\breve{p}, 0)$. Then, by Theorem 3, $E(\tilde{\breve{\Gamma}}) = E(\breve{\Gamma})$. Let $\tilde{\Gamma}$ be the game obtained by replacing in Γ the price function p by $\max(p,0)$. Again by Theorem 3, $E(\tilde{\Gamma}) = E(\Gamma)$. But $\tilde{\Gamma} = \tilde{\breve{\Gamma}}$. Thus $\#E(\Gamma) \leq 1$. $\square$

5.2. Equilibrium Semi-uniqueness result of Gaudet-Salant

The result in Gaudet and Salant(1991) we are interested in is the following equilibrium semi-uniqueness result.

Theorem 9. *Consider a homogeneous Cournot oligopoly where:*

- *a. no firm has a capacity constraint;*
- *b. p has a market satiation point $v \in]0,+\infty[$ and $p(y) = 0$ $(y \geq v)$;*
- *c. the price function p is decreasing;*
- *d. p is continuous and $p \restriction [0,v[$ is twice continuously differentiable;*
- *e. each cost function c_i is twice continuously differentiable;*
- *f. each cost function c_i is strictly increasing, even $Dc_i(x_i) > 0$ $(x_i > 0)$;*
- *g. for every i and $y \in [0,v[$ there exists $\alpha < 0$ such that $Dp(y) - D^2c_i \leq \alpha$;*
- *h. for each Cournot equilibrium $\mathbf{e}$*

$$\sum_{k \in \{j \in N \mid e_j > 0\}} -\frac{D^2p(\underline{\mathbf{e}})e_k + Dp(\underline{\mathbf{e}})}{Dp(\underline{\mathbf{e}}) - D^2c_k(e_k)} < 1. \quad (22)$$

Then there exists at most one Cournot equilibrium. $\diamond$

First note that in Theorem 9 by Proposition 8(3) for each equilibrium $\mathbf{e}$ it holds that $\underline{\mathbf{e}} < v$ and that therefore $Dp(\mathbf{e})$ in condition h makes sense. Also note that with $t_i : \mathbb{R}_+ \times [0, v[$ defined by $t_i(x_i, y) = Dp(y)x_i + p(y) - Dc_i(x_i)$, (22) becomes: for each equilibrium $\mathbf{e}$

$$\sum_{k \in \{j \in N \mid e_j > 0\}} -\frac{D_2 t_i(e_i; \underline{\mathbf{e}})}{D_1 t_i(e_i; \underline{\mathbf{e}})} < 1.$$

If in Theorem 9 in addition, for every $y \in [0, v[$, the (marginal revenue) condition $yD^2p(y) + Dp(y) \leq 0$ holds, then Theorem 4 guarantees that conditional profit functions are quasi-concave and Theorem 5 implies that there exists a unique equilibrium.

The proof of Theorem 9 in Gaudet and Salant(1991) is based on an analysis of the backward best reply correspondences. Theorem 9 is a variant of a result in Kolstad and Mathiesen(1987). It improves upon this result (but does not imply it) by not excluding degenerate equilibria. The proof given in Gaudet and Salant(1991) is much more elementary than the proof in Kolstad and Mathiesen(1987) which deals with equilibria as the solution of a complementarity problem to which differential topological fixed point index theory is applied.

It is good to note that Theorem 9 does not imply the simple result in Corollary 2. The main obstruction for this seems to be the condition in h which is a strong variant of the first Fisher-Hahn condition. It would be interesting to have an improvement of Theorem 9 that implies results like Corollary 2 (that is, results that allow, e.g. capacity constraints). As the main objects in Theorem 9 are the marginal reductions t_i, a variant for aggregative games should be possible.

That Theorem 9 is quite compatible with log-concave price functions, is clear from Proposition 12(1).

Clearly Theorem 9 does not imply Theorem 7 because Theorem 7 allows never vanishing price functions as $1/(y+1)$. This point is quite straightforward. Note, however, that Theorem 9 does not imply Theorem 8 even when p is assumed to be twice continuously differentiable on its positive support and cost functions are assumed continuously twice differentiable over their whole domain. For instance, it can be checked that the duopoly without capacity constraints, strictly convex cost functions

$$c_1(x_1) = x_1 + \frac{1}{100}(x_1 - 2)^4 - \frac{16}{100}, \;\; c_2(x_2) = 1000\left((x_2+1)^2 - 1\right),$$

and price function (note that $r \upharpoonright [0, v]$ is concave)

$$p(y) = \begin{cases} y^2 - 6y + 13 & \text{if } y \in [0,2], \\ \frac{8}{y} + 1 & \text{if } y \in [2,3], \\ \max\left(0, \frac{8}{y} + 1 - \frac{(y-3)^3}{y}\right) & \text{if } y \in [3, +\infty[, \end{cases}$$

satisfies the conditions of Theorem 7 and has a unique equilibrium, which is $\mathbf{e} = (2, 0)$. However, this duopoly does not satisfy the conditions of Theorem 9 because

$$\sum_{k \in \{j \in N \mid e_j > 0\}} -\frac{D^2p(\underline{\mathbf{e}})\, e_k + Dp(\underline{\mathbf{e}})}{Dp(\underline{\mathbf{e}}) - D^2c_k(e_k)} = -\left(\frac{2 \cdot 2 - 2}{-2 - 0}\right) = 1.$$

Besides, it is good to note, that in general the proof of Gaudet and Salant(1991) cannot be adapted to oligopolies with cost functions that are not twice differentiable because of its very formulation.

5.3. Equilibrium Uniqueness Result of Szidarovszky-Okuguchi

Oligopolies with price functions that are unbounded at 0 lead to special interesting complications. In Szidarovszky and Okuguchi(1997) the following was proven:

Theorem 10. *Consider a homogeneous Cournot oligopoly with at least two firms. Suppose no firm has a capacity constraint, each cost function c_i is twice differentiable with $c_i(0) = 0$, $Dc_i > 0$, $D^2c_i > 0$ and $p(y) = d/y$ $(y > 0)$ where $d > 0$.*[8] *Then there exists a unique equilibrium.* $\diamond$

This result was obtained by analysing the backward best reply correspondences. As payoff functions are not continuous at zero, standard existence results like that of Nikaido-Isoda cannot be used.

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[8] Remember that the value of p at 0 is not important. So the value of $p(0)$ can be chosen arbitrarily.

Generalized Proportional Solutions to Games with Restricted Cooperation

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Abstract In TU–cooperative game with restricted cooperation the values of characteristic function $v(S)$ are defined only for $S \in \mathcal{A}$, where $\mathcal{A}$ is a collection of some nonempty coalitions of players. If $\mathcal{A}$ is a set of all singletones, then a claim problem arises, thus we have a claim problem with coalition demands.
We examine several generalizations of the Proportional method for claim problems: the Proportional solution, the Weakly Proportional solution, the Proportional Nucleolus, and g–solutions that generalize the Weighted Entropy solution. We describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Weakly Proportional solution. The necessary and sufficient condition on $\mathcal{A}$ for coincidence g–solution and the Weakly Proportional solution and sufficient condition for coincidence all g–solutions and the Proportional Nucleolus are obtained.

Keywords: claim problem, cooperative games, proportional solution, weighted entropy, nucleolus.

1. Introduction

A *TU–cooperative game with restricted cooperation* is a quadruple $(N, \mathcal{A}, c, v)$, where N is a finite set of agents, $\mathcal{A}$ is a collection of nonempty coalitions of agents, c is a positive real number (the amount of resourses to be divided by agents), $v = \{v(T)\}_{T \in \mathcal{A}}$, where $v(T) > 0$ is a claim of coalition T. We assume that $\mathcal{A}$ covers N and $N \notin \mathcal{A}$.

A *set of imputations of* $(N, \mathcal{A}, c, v)$ is the set

$$\{\{y_i\}_{i \in N} : \ y_i \geq 0, \ \sum_{i \in N} y_i = c\}.$$

A *solution* F is a map that associates to any game $(N, \mathcal{A}, c, v)$ a subset of its set of imputations. We denote $y(S) = \sum_{i \in S} y_i$.

If $\mathcal{A} = \{\{i\} : i \in N\}$ then a *claim problem* arises, therefore, a cooperative game with restricted cooperation can be considered as a claim problem with coalition demands.

Solutions of claim problem and their axiomatic justifications are described in surveys (Moulin, 2002) and (Thomson, 2003). For games with restricted cooperation, several generalizations of well known Proportional solution and Uniform Losses solution for claim problem are examined in (Naumova, 2011). In particular, she considers the Proportional Nucleolus, the Weighted Entropy solution, and the Weakly

Proportional solution, where the ratios of total shares of coalitions to their claims are equal for disjoint coalitions in $\mathcal{A}$. Necessary and sufficient condition on $\mathcal{A}$ for coincidence the Weighted Entropy solution and the Weakly Proportional solution, necessary condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution, and necessary condition on $\mathcal{A}$ for inclusion the Weighted Entropy solution in the Weakly Proportional solution are obtained in that paper.

In this paper we consider generalizations of Weighted Entropy solution called g–solutions. For TU-cooperative games with positive characteristic function, i.e., for the case $\mathcal{A} = 2^N \setminus \{\emptyset\}$, these solutions are defined and axiomatically justified in (Yanovskaya, 2002). For each g, the condition on $\mathcal{A}$ for coincidence g–solution with the Weakly Proportional solution is the same as for the case, where g–solution is the Weighted Entropy solution. Sufficient condition for coincidence all g–solutions and the Proportional Nucleolus is obtained. Moreover, we describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Weakly Proportional solution.

The paper is organized as follows. The definitions of several generalizations of the Proportional solution and conditions on $\mathcal{A}$ for existence the Proportional and the Weakly Proportional solutions are described in Section 2. Some properties of g–solutions that will be used in next sections are obtained in Section 3. Necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Proportional solution is obtained in Section 4. In Section 5 we describe necessary and sufficient condition on $\mathcal{A}$ for inclusion the Proportional Nucleolus in the Weakly Proportional solution and necessary and sufficient condition on $\mathcal{A}$ for inclusion g–solution in the Weakly Proportional solution. In Section 6 we describe conditions on $\mathcal{A}$ for coincidence g–solution with the Weakly Proportional solution and for coincidence all g–solutions with the Proportional Nucleolus.

2. Generalizations of the Proportional solution

Definition 1. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Proportional solution* of $(N, \mathcal{A}, c, v)$ iff there exists $\alpha > 0$ such that $y(T) = \alpha v(T)$ for all $T \in \mathcal{A}$.

Definition 2. An imputation $y = \{y_i\}_{i\in N}$ belongs to the *Weakly Proportional solution* of $(N, \mathcal{A}, c, v)$ ($y \in \mathcal{WP}(N, \mathcal{A}, c, v)$) iff $y(S)/v(S) = y(Q)/v(Q)$ for all $S, Q \in \mathcal{A}$ with $S \cap Q = \emptyset$.

The following results of the author will be used in this paper.

Proposition 1 (Naumova, 2011, Theorem 1.). *The Proportional solution of $(N, \mathcal{A}, c, v)$ is nonempty for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a minimal covering of N.*

A set of coalitions $\mathcal{A}$ generates the undirected graph $G = G(\mathcal{A})$, where $\mathcal{A}$ is the set of nodes and $K, L \in \mathcal{A}$ are adjacent iff $K \cap L = \emptyset$.

Theorem 1 (Naumova, 2011, Theorem 3.). *The Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ is a nonempty set for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ satisfies the following condition.*
C0. If a single node is taken out from each component of $G(\mathcal{A})$, then the remaining elements of $\mathcal{A}$ do not cover N.

Definition 3. Let $X \subset R^n$, $u_1, \dots, u_k$ be functions defined on X. For $z \in X$, let π be a permutation of $\{1, \dots, k\}$ such that $u_{\pi(i)}(z) \le u_{\pi(i+1)}(z)$, $\theta(z) = \{u_{\pi(i)}(z)\}_{i=1}^k$. Then $y \in X$ belongs to the *nucleolus with respect to* $u_1, \dots, u_k$ *on* X iff

$$\theta(y) \geq_{lex} \theta(z) \quad \text{for all} \quad z \in X.$$

Definition 4. A vector $y = \{y_i\}_{i \in N}$ belongs to the *Proportional Nucleolus* of $(N, \mathcal{A}, c, v)$ iff y belongs to the nucleolus w.r.t. $\{u_T\}_{T \in \mathcal{A}}$ with $u_T(z) = z(T)/v(T)$ on the set of imputations of $(N, \mathcal{A}, c, v)$.

For each $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ is nonempty and defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.

Let $\mathcal{G}$ be a class of strictly increasing continuos functions g defined on $(0, +\infty)$ such that $g(1) = 0$, and $\lim\limits_{x \to 0} \int\limits_a^x g(t)dt < +\infty$ for each $a > 0$.

Definition 5. Let $g \in \mathcal{G}$, $f(z) = \sum\limits_{Q \in \mathcal{A}} \int\limits_{v(Q)}^{z(Q)} g(t/v(Q))dt$. A vector $y = \{y_i\}_{i \in N}$ belongs to *g–solution* of $(N, \mathcal{A}, c, v)$ iff y minimizes f on the set of imputations of $(N, \mathcal{A}, c, v)$.

For each $g \in G$, g–solution of $(N, \mathcal{A}, c, v)$ is a nonempty set because f is a continuos function on the set of imputations.

For $\mathcal{A} = 2^N \setminus \{\emptyset\}$, g–solutions are described in (Yanovskaya, 2002). For each $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the g–solution of $(N, \mathcal{A}, c, v)$ is nonempty.

Examples of g–solutions

1. Let $g(t) = \ln t$, then $\int\limits_{v(S)}^{z(S)} g(t/v(S))dt = z(S)[\ln(z(S)/v(S)) - 1] + v(S)$ and the g–solution is the *Weighted Entropy solution* (Naumova, 2008, 2011).

2. Let $g(t) = t^q - 1$, where $q > 0$, then we obtain the minimization problem for $\sum\limits_{S \in \mathcal{A}} z(S)[\frac{z(S)^q}{(q+1)v(S)^q} - 1]$ that was considered for $\mathcal{A} = 2^N \setminus \{\emptyset\}$ in (Yanovskaya, 2002).

3. Properties of g–solutions

Property 1. *Let* $g \in G$, $\lim\limits_{t \to 0} g(t) = -\infty$, *and* x *belong to* g*–solution of* $(N, \mathcal{A}, c, v)$. *Then* $x(S) > 0$ *for all* $S \in \mathcal{A}$.

Proof. Suppose that there exist $(N, \mathcal{A}, c, v)$, $S \in \mathcal{A}$, and x in g–solution of $(N, \mathcal{A}, c, v)$ such that $x(S) = 0$. Let $0 < \epsilon < \min\{x_k : x_k > 0\}$. Let

$$M = \max_{T: T \in \mathcal{A}, x(T) > 0} \ \max_{t \in [x(T) - \epsilon, x(T) + \epsilon]} |g(t/v(T))|.$$

Fix $\delta > 0$ such that $\delta < \min\{\epsilon, \min_{T \in \mathcal{A}} v(T)\}$ and $|g(\delta/v(S))| > 2^{|N|}M$. Let $i \in S$, $j \in N$, $x_j > 0$.

Take $z \in R^{|N|}$ such that $z_i = x_i + \delta$, $z_j = x_j - \delta$, $z_k = x_k$ for $k \neq i, j$. Then $\sum\limits_{T\in\mathcal{A}} \int\limits_{v(T)}^{z(T)} g(t/v(T))dt - \sum\limits_{T\in\mathcal{A}} \int\limits_{v(T)}^{x(T)} g(t/v(T))dt =$

$\sum\limits_{T\in\mathcal{A}: i\in T, j\notin T} \int\limits_{x(T)}^{x(T)+\delta} g(t/v(T))dt - \sum\limits_{T\in\mathcal{A}: i\notin T, j\in T} \int\limits_{x(T)-\delta}^{x(T)} g(t/v(T))dt.$

If $i \notin T$, $j \in T$ then $|\int\limits_{x(T)-\delta}^{x(T)} g(t/v(T))dt| \leq \delta M$.

If $T = S$ then $\int\limits_{x(S)}^{x(S)+\delta} g(t/v(S))dt = \int\limits_{0}^{\delta} g(t/v(S))dt < -2^{|N|}M\delta$.

If $i \in T$, $j \notin T$, $x(T) = 0$, then $\int\limits_{x(T)}^{x(T)+\delta} g(t/v(T))dt < 0$ since $\delta < v(T)$.

If $i \in T$, $j \notin T$, $x(T) > 0$, then $|g(t/v(T))| \leq M$ as $t \in [x(T), x(T)+\delta]$, hence $|\int\limits_{x(T)}^{x(T)+\delta} g(t/v(T))dt| \leq \delta M$.

Thus,

$$\sum_{T\in\mathcal{A}} \int_{v(T)}^{z(T)} g(t/v(T))dt - \sum_{T\in\mathcal{A}} \int_{v(T)}^{x(T)} g(t/v(T))dt < (|\mathcal{A}|-1)\delta M - 2^{|N|}M\delta < 0$$

and x is not in g–solution of $(N, \mathcal{A}, c, v)$. □

Property 2. *For each $g \in \mathcal{G}$, $f(z) = \sum\limits_{Q\in\mathcal{A}} \int\limits_{v(Q)}^{z(Q)} g(t/v(Q))dt$ is a convex function of z and for all $\mathcal{A}$, $c > 0$, v with $v(T) > 0$, the g–solution of $(N, \mathcal{A}, c, v)$ defines uniquely total amounts $y(T)$ for each $T \in \mathcal{A}$.*

Proof. Let $g \in \mathcal{G}$, $a > 0$, $\psi(q) = \int\limits_{a}^{q} g(t)dt$ for $q \geq 0$. If $g \in \mathcal{G}$ and $\lim\limits_{t\to 0} g(t) > -\infty$, then $\psi(q)$ is a strictly convex function on $[0, +\infty)$. If $\lim\limits_{t\to 0} g(t) = -\infty$, then $\psi(q)$ is a convex function on $[0, +\infty)$ and a strictly convex function on $(0, +\infty)$. Therefore $f(z)$ is a convex function of z and in view of Property 1, if y and z belong to g–solution of $(N, \mathcal{A}, c, v)$, then $y(T) = z(T)$ for all $T \in \mathcal{A}$. □

Property 3. *For each x in g–solution of $(N, \mathcal{A}, c, v)$, $x_i > 0$ implies*

$$\sum_{T\in\mathcal{A}: i\in T} g(x(T)/v(T)) \leq \sum_{T\in\mathcal{A}: j\in T} g(x(T)/v(T)) \quad \textit{for all} \quad j \in N. \tag{1}$$

Proof. Note that in view of Property 1, $g(x(Q)/v(Q))$ is well defined for all $Q \in \mathcal{A}$. Let $x_i > 0$. Suppose that there exists $j \in N$ such that

$$\sum_{T\in\mathcal{A}: j\in T} g(x(T)/v(T)) < \sum_{T\in\mathcal{A}: i\in T} g(x(T)/v(T)).$$

Consider $\epsilon \geq 0$ and $y(\epsilon) \in R^{|N|}$ such that $\epsilon < x_i$, $y(\epsilon)_i = x_i - \epsilon$, $y(\epsilon)_j = x_j + \epsilon$, $y(\epsilon)_k = x_k$ for $k \neq i, j$. Let

$$F(\epsilon) = \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{y(\epsilon)(Q)} g(t/v(Q))dt - \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{x(Q)} g(t/v(Q))dt,$$

then

$$F(\epsilon) = \sum_{Q \in \mathcal{A}: i \in Q, j \notin Q} \int_{x(Q)}^{x(Q)-\epsilon} g(t/v(Q))dt + \sum_{Q \in \mathcal{A}: i \notin Q, j \in Q} \int_{x(Q)}^{x(Q)+\epsilon} g(t/v(Q))dt,$$

$$F'(0) = -\sum_{Q \in \mathcal{A}: i \in Q, j \notin Q} g(x(Q)/v(Q)) + \sum_{Q \in \mathcal{A}: i \notin Q, j \in Q} g(x(Q)/v(Q)) < 0.$$

Hence, $F(\epsilon) < 0$ for some $\epsilon > 0$ and x does not belong to g–solution of $(N, \mathcal{A}, c, v)$. □

Property 4. *Let $g \in \mathcal{G}$ and x be an imputation of $(N, \mathcal{A}, c, v)$ such that $x_i > 0$ implies (1). Then x belongs to g–solution of $(N, \mathcal{A}, c, v)$.*

Proof. For each imputation z of $(N, \mathcal{A}, c, v)$, let $f(z) = \sum_{Q \in \mathcal{A}} \int_{v(Q)}^{z(Q)} g(t/v(Q))dt$.

If $z_j > 0$ for all $j \in N$ then f is differentiable at z and

$$\frac{\partial}{\partial z_j} f(z) = \sum_{T \in \mathcal{A}:\, T \ni j} g(z(T)/v(T)). \tag{2}$$

If z and w are imputations of $(N, \mathcal{A}, c, v)$ such that $z_j, w_j > 0$ for all $j \in N$, then, in view of Property 2,

$$f(w) - f(z) \geq \sum_{j \in N} \frac{\partial f(z)}{\partial z_j}(w_j - z_j). \tag{3}$$

Note that if $x_i > 0$ then for all $Q \ni i$, $x(Q) > 0$ and $g(x(Q)/v(Q))$ is well defined. Hence, in view of (1), for all $j \in N$, $\sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T))$ is well defined. Let y be an imputation of $(N, \mathcal{A}, c, v)$. There exist imputations z^k and w^k with positive coordinates such that $\lim_{k \to +\infty} z^k = x$, $\lim_{k \to +\infty} w^k = y$, then it follows from (3) and (2) that

$$f(y) - f(x) \geq \sum_{j \in N} (y_j - x_j) \sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T)). \tag{4}$$

Let $x_i > 0$, then (1) implies

$$\sum_{j \in N} x_j \sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T)) = c \sum_{T \in \mathcal{A}:\, T \ni i} g(x(T)/v(T)), \tag{5}$$

$$\sum_{j \in N} y_j \sum_{T \in \mathcal{A}:\, T \ni j} g(x(T)/v(T)) \geq c \sum_{T \in \mathcal{A}:\, T \ni i} g(x(T)/v(T)). \tag{6}$$

It follows from (4), (5), (6) that $f(y) - f(x) \geq 0$, i.e., x belongs to g–solution of $(N, \mathcal{A}, c, v)$. □

4. When generalized proportional solutions are proportional?

Proposition 2 (Naumova, 2011, Proposition 1). *The Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a partition of N.*

Proposition 3. *For each $g \in \mathcal{G}$, g–solution is contained in the Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is a partition of N.*

Proof. Let $\mathcal{A}$ be a partition of N. Then for all $S \in \mathcal{A}$, $i \in S$, all imputations x of $(N, \mathcal{A}, c, v)$,

$$\sum_{T \in \mathcal{A}:\, T \ni i} g(x(T)/v(T)) = g(x(S)/v(S)). \tag{7}$$

If x belongs to the Proportional solution then by (7) and Property 4, x belongs to g–solution. Since in the considered case $x(S)$ are defined uniquely for all $S \in \mathcal{A}$ and g–solution depends only on $x(S)$ for all $S \in \mathcal{A}$, the Proportional solution coincides with g–solution.

Let g–solution be always contained in the Proportional solution. Suppose that $\mathcal{A}$ is not a partition of N, then there exist $P, Q \in \mathcal{A}$ such that $P \cap Q \neq \emptyset$. We take the following v: $v(P) = 2$, $v(T) = \epsilon$ otherwise, where $\epsilon < 1/|N|$.

Let x belong to g–solution of $(N, \mathcal{A}, 1, v)$. Since x is proportional, $x(T) = \epsilon x(P)/2 \leq \epsilon/2$ for all $T \in \mathcal{A} \setminus \{P\}$, hence $x_i \leq \epsilon/2$ for all $i \in N \setminus P$. If $x_i \leq \epsilon$ for all $i \in P$, then $x(N) \leq \epsilon|N| < 1$, hence there exists $j_0 \in P \setminus \cup_{T \in \mathcal{A} \setminus \{P\}} T$ such that $x_{j_0} > \epsilon$. Let $i_0 \in P \cap Q$. By Property 3,

$$g(x(P)/v(P)) \leq \sum_{T \in \mathcal{A}:\, T \ni i_0} g(x(T)/v(T)).$$

Since $x(T)/v(T) \leq 1/2$ for all $T \in \mathcal{A}$, this contradicts $g(1) = 0$. Hence $\mathcal{A}$ is a partition of N. $\square$

5. When generalized proportional solutions are weakly proportional?

For $i \in N$, denote $\mathcal{A}_i = \{T \in \mathcal{A} : i \in T\}$.

Definition 6. A collection of coalitions $\mathcal{A}$ is *weakly mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where

C1) each $\mathcal{B}^i$ is contained in a partition of N;
C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C3) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Remark 1. If $k \leq 2$ then C3 follows from C1 and C2.

Remark 2. If $\mathcal{A}$ is weakly mixed then it satisfies the condition C0 of Theorem 1.

Proof. Let $\mathcal{A}$ be weakly mixed at N. Take $j_0 \in N$ such that $|\mathcal{A}_{j_0}| \geq |\mathcal{A}_i|$ for all $i \in N$. Let $\mathcal{A}_{j_0} = \{Q_t\}_{t \in M}$, where $Q_t \in \mathcal{B}^t$, $M \subset \{1, \ldots, k\}$.

Let $S_t \in \mathcal{B}^t$ for all $t \leq k$. Since $\mathcal{A}$ is weakly mixed, there exists $i_o \in \bigcap_{t \in M} S_t$. In view of definition of j_0, $\mathcal{A}_{i_0} = \{S_t : t \in M\}$. Therefore, if for each $t \in \{1, \ldots, k\}$, S_t is taken out from $\mathcal{A}$, then the remaining elements of $\mathcal{A}$ do not cover i_0. $\square$

Example 1. Let $N = \{1, 2, \ldots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N.

Example 2. $N = \{1, 2, \ldots, 12\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$,
$\mathcal{B}^2 = \{\{3, 5, 9, 10\}, \{4, 6, , 11, 12\}\}$,
$\mathcal{B}^3 = \{\{1, 7, 9, 11\}, \{2, 8, 10, 12, 13\}\}$.
Then $\mathcal{A}$ is weakly mixed at N.

Example 3. Let $N = \{1, 2, \ldots, 6\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathcal{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
$\mathcal{B}^3 = \{\{1, 4, 5\}, \{2, 3, 6\}\}$,
then $\mathcal{C}$ satisfies C0, C1, and C2, but does not satisfy C3 (for $i = 1$ and $Q = \{1, 2\}$), hence $\mathcal{C}$ is not weaky mixed at N.

Proposition 4 (Naumova, 2011, Proposition 3.). *Let the Proportional Nucleolus of $(N, \mathcal{A}, t, v)$ be contained in the Weakly Proportional solution of $(N, \mathcal{A}, t, v)$ for all $t > 0$, all v with $v(T) > 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Theorem 2. *The Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ is contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is weakly mixed at N.*

Proof. Let $\mathcal{A}$ be weakly mixed at N and x belong to the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$. Suppose that x is not weakly proportional, i.e., there exist $S, Q \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $x(Q)/v(Q) < x(S)/v(S)$. Then there exists $i_0 \in S$ such that $x_{i_0} > 0$. Since $\mathcal{A}$ is weakly mixed, there exists $j \in N$ such that $\mathcal{A}_j \supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Take $\delta > 0$ such that

$$(x(Q) + \delta)/v(Q) < (x(S) - \delta)/v(S)$$

and $\delta < x_{i_0}$. Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} - \delta$, $y_j = x_j + \delta$, $y_t = x_t$ otherwise. Then $y(P) < x(P)$ only for $P = S$ and $y(Q) > x(Q)$. Since $y(Q)/v(Q) < y(S)/v(S)$, this contradicts the definition of the Proportional Nucleolus.

Let the Proportional Nucleolus be always contained in the Weakly Proportional solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 1. Then in view of Proposition 4 and Theorem 1, $\mathcal{A}$ satisfies C1 and C2. Suppose that $\mathcal{A}$ is not weakly mixed. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ such that $S \cap Q = \emptyset$ and $\mathcal{A}_j \not\supset \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ for all $j \in N$. Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(S) = 1$,
$v(P) = |N|^2$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.
Let x belong to the Proportional Nucleolus and to the Weakly Proportional solution of $(N, \mathcal{A}, 1, v)$. Since x is weakly proportional and $v(S) + v(Q) > 1$ we have $x(Q) < v(Q) = \epsilon$. There exists $j_0 \in N$ such that $x_{j_0} \geq 1/|N|$. Then $j_0 \notin Q$ and $j_0 \neq i_0$.

Take $\delta > 0$ such that $\delta < 1/|N|$ and for each $T, P \in \mathcal{A}$,

$$x(T)/v(T) < x(P)/v(P) \quad \text{implies} \quad (x(T)+\delta)/v(T) < (x(P)-\delta)/v(P).$$

Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} + \delta$, $y_{j_0} = x_{j_0} - \delta$, $y_i = x_i$ otherwise.

We prove that $x(P)/v(P) < x(T)/v(T)$ for some $P \in \mathcal{A}$ with $y(P) > x(P)$ and all $T \in \mathcal{A}$ with $y(T) < x(T)$ and this would imply that x does not belong to the Proportional Nucleolus of $(N, \mathcal{A}, 1, v)$. Consider 2 cases.

Case 1. $j_0 \notin S$. Let $y(T) < x(T)$, then $T \ni j_0$ and $v(T) = \epsilon$, hence $x(T)/v(T) \geq x_{j_0}/\epsilon > 1$. Since $x(Q)/v(Q) < 1$ and $y(Q) > x(Q)$, x does not belong to the Proportional Nucleolus of $(N, \mathcal{A}, 1, v)$ in this case.

Case 2. $j_0 \in S$, then $\mathcal{A}_{i_0} \setminus \mathcal{A}_{j_0} \setminus \{Q\} \ni P$, where $x(P)/v(P) \leq 1/(|N|^2)$ and $y(P) > x(P)$. If $y(T) < x(T)$ then either $T = S$ and $x(S)/v(S) \geq 1/|N| > 1/(|N|^2)$ or $v(T) = \epsilon$ and $x(T)/v(T) \geq x_{j_0}/\epsilon > 1$. Thus, x does not belong to the Proportional Nucleolus of $(N, \mathcal{A}, 1, v)$ in this case. □

Definition 7. A collection of coalitions $\mathcal{A}$ is *mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where
C1) each $\mathcal{B}^i$ is contained in a partition of N;
C2) $Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \neq j$ imply $Q \cap S \neq \emptyset$;
C4) for each $i \in N$, $Q \in \mathcal{A}_i$, $S \in \mathcal{A}$ with $Q \cap S = \emptyset$, there exists $j \in N$ such that $\mathcal{A}_j = \mathcal{A}_i \cup \{S\} \setminus \{Q\}$.

Note that if $\mathcal{A}$ is mixed at N then $\mathcal{A}$ is weakly mixed at N.

Example 4. If $\mathcal{A}$ is weakly mixed at N and all $i \in N$ belong to the same number of coalitions, then $\mathcal{A}$ is mixed at N.

Example 5. Let $N = \{1, 2, \ldots, 6\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{A}$ is mixed at N.

Example 6. Let $N = \{1, 2, \ldots, 5\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{C}$ is weakly mixed at N but not mixed at N. (For $i = 3$, the condition C4 is not realized.)

Proposition 5. *Let g–solution of $(N, \mathcal{A}, c, v)$ be contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, all v with $v(T) > 0$. Then the case $P, Q, S \in \mathcal{A}$, $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$ is impossible.*

Proof. Suppose that there exist $P, Q, S \in \mathcal{A}$ such that $P \neq Q$, $P \cap S = Q \cap S = \emptyset$, $P \cap Q \neq \emptyset$. Let $i_0 \in P \cap Q$, $\mathcal{A}_0 = \{T \in \mathcal{A} : i_0 \in T, T \cap S \neq \emptyset\}$.

Let $0 < \epsilon < 1/(4|N|)$. We take the following v:
$v(T) = 1$ for $T \in \mathcal{A}_0 \cup \{P, S\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to g–solution of $(N, \mathcal{A}, 1, v)$. Since x is weakly proportional and $S \cap P = \emptyset$, we have $x(S) = x(P) \leq 1/2$. Then $x(Q)/v(Q) = x(S)/v(S) \leq 1/2$. As $v(Q) = \epsilon$, $x(Q) \leq \epsilon/2$. Consider 2 cases.

Case 1. $x_i \le \epsilon$ for all $i \in P$. Then $x(S \cup P) = x(S) + x(P) \le 2\epsilon|P| < 1/2$. If $x_i \le \epsilon$ for all $i \in N\backslash(P \cup S)$ then $x(N) < 3/4$, hence $x_{j_0} > \epsilon$ for some $j_0 \in N\backslash(P \cup S)$.

Case 2. There exists $j_0 \in P$ with $x_{j_0} > \epsilon$.

In both cases, $x_{j_0} > \epsilon$, $j_0 \notin S$, $j_0 \notin Q$ because $x(Q) \le \epsilon/2$, hence $j_0 \neq i_0$. Let $j_0 \in T$, $i_0 \notin T$. Then $T \notin \mathcal{A}_0 \cup \{P, S\}$, hence $v(T) = \epsilon$ and $x(T)/v(T) > 1$. Thus

$$\sum_{T \in \mathcal{A}:\, T \ni j_0, T \not\ni i_0} g(x(T)/v(T)) \ge 0. \tag{8}$$

Let $j_0 \notin T$, $i_0 \in T$. If $v(T) = \epsilon$ then $T \cap S = \emptyset$ and $x(T)/v(T) = x(S)/v(S) \le 1/2$. If $v(T) = 1$, then $v(T) \ge x(T)$. Therefore

$$\sum_{T \in \mathcal{A}:\, T \not\ni j_0, T \ni i_0} g(x(T)/v(T)) \le g(x(Q)/v(Q)) < 0. \tag{9}$$

It follows from (8) and (9) that

$$\sum_{T \in \mathcal{A}:\, T \ni j_0} g(x(T)/v(T)) > \sum_{T \in \mathcal{A}:\, T \ni i_0} g(x(T)/v(T)),$$

but this contradicts Property 3. □

Theorem 3. *Let $g \in \mathcal{G}$. The g–solution of $(N, \mathcal{A}, c, v)$ is contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is mixed at N.*

Proof. Let $\mathcal{A}$ be a mixed collection of coalitions. Let x belong to g–solution of $(N, \mathcal{A}, c, v)$. Suppose that x does not belong to the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$, i.e., there exist $Q, S \in \mathcal{A}$ such that $Q \cap S = \emptyset$ and $x(Q)/v(Q) > x(S)/v(S)$. There exists $i_0 \in Q$ with $x_{i_0} > 0$. Since $\mathcal{A}$ is mixed, there exists $j_0 \in N$ such that $\mathcal{A}_{j_0} = \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$ $\{T \in \mathcal{A} : j_0 \in T\} = \{S\} \cup \{P_i\}_{i=1}^k$. Then

$$\sum_{T \in \mathcal{A}:\, T \not\ni j_0, T \ni i_0} g(x(T)/v(T)) = g(x(Q)/v(Q)),$$

$$\sum_{T \in \mathcal{A}:\, T \ni j_0, T \not\ni i_0} g(x(T)/v(T)) = g(x(S)/v(S)),$$

hence

$$\sum_{T \in \mathcal{A}:\, T \ni i_0} g(x(T)/v(T)) > \sum_{T \in \mathcal{A}:\, T \ni j_0} g(x(T)/v(T)),$$

but this contradicts Property 3. Thus, x belongs to the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$.

Let g–solution be always contained in the Weakly Proportional solution. Let $\mathcal{B}^i$ be components of the graph $G(\mathcal{A})$ used in Theorem 1. Then in view of Proposition 5 and Theorem 1, $\mathcal{A}$ satisfies C1 and C2. Suppose that $\mathcal{A}$ is not mixed at N. Then there exist $i_0 \in N$, $Q \in \mathcal{A}_{i_0}$, and $S \in \mathcal{A}$ with $S \cap Q = \emptyset$ such that for each $j \in N$, $\mathcal{A}_j \neq \mathcal{A}_{i_0} \cup \{S\} \setminus \{Q\}$. Let $0 < \epsilon < 1/|N|$. We take the following v:
$v(S) = 1$,
$v(P) > 1$ for $P \in \mathcal{A}_{i_0} \setminus \{Q\}$,
$v(T) = \epsilon$ otherwise.

Let x belong to g–solution and to the Weakly Proportional solution of $(N,\mathcal{A},1,v)$. Since x is weakly proportional and $v(S)+v(Q)>1$ we have $x(Q)<v(Q)=\epsilon$. There exists $j_0\in N$ such that $x_{j_0}\geq 1/|N|$. Then $j_0\notin Q$. We shall prove that

$$\sum_{T\in\mathcal{A}:T\ni i_0} g(x(T)/v(T)) < \sum_{T\in\mathcal{A}:T\ni j_0} g(x(T)/v(T)), \tag{10}$$

and this contradicts Property 3.

The following 3 cases are possible.

1. $j_0\notin S$.
2. $j_0\in S$, $\mathcal{A}_{i_0}\setminus\{Q\}\neq\emptyset$, and $j_0\notin\bigcap_{P\in\mathcal{A}_{i_0}\setminus\{Q\}} P$.
3. $j_0\in S$, $j_0\in S\cap\bigcap_{P\in\mathcal{A}_{i_0}\setminus\{Q\}} P$, and there exists $T_0\in\mathcal{A}\setminus\mathcal{A}_{i_0}$ such that $j_0\in T_0$.

Case 1.

$$\sum_{T\in\mathcal{A}:\,T\not\ni j_0,T\ni i_0} g(x(T)/v(T)) \leq g(x(Q)/v(Q)) < 0.$$

Since $j_0\notin S$, $x(T)>v(T)=\epsilon$ for all $T\in\mathcal{A}_{j_0}\setminus\mathcal{A}_{i_0}$, therefore,

$$\sum_{T\in\mathcal{A}:\,T\ni j_0,T\not\ni i_0} g(x(T)/v(T)) \geq 0,$$

this implies (10).

Case 2. Since $\mathcal{A}_{i_0}\setminus\mathcal{A}j_0\setminus\{Q\}\neq\emptyset$,

$$\sum_{T\in\mathcal{A}:\,T\not\ni j_0,T\ni i_0} g(x(T)/v(T)) =$$

$$g(x(Q)/v(Q)) + \sum_{T\in\mathcal{A}_{i_0}\setminus\mathcal{A}j_0\setminus\{Q\}} g(x(T)/v(T)) < g(x(Q/v(Q)).$$

If $T\in\mathcal{A}_{j_0}\setminus\mathcal{A}_{i_0}$ then either $T=S$ or $x(T)>v(T)=\epsilon$, therefore

$$\sum_{T\in\mathcal{A}:\,T\ni j_0,T\not\ni i_0} g(x(T)/v(T)) \geq g(x(S)/v(S)).$$

Since $x(Q)/v(Q)=x(S)/v(S)$, we obtain (10).

Case 3.

$$\sum_{T\in\mathcal{A}:\,T\not\ni j_0,T\ni i_0} g(x(T)/v(T)) = g(x(Q)/v(Q)),$$

$$\sum_{T\in\mathcal{A}:\,T\ni j_0,T\not\ni i_0} g(x(T)/v(T)) \geq g(x(S)/v(S)) + g(x(T_0)/v(T_0)) > g(x(S)/v(S).$$

Since $x(Q)/v(Q)=x(S)/v(S)$, we obtain (10). □

6. When different generalizations give the same result?

Definition 8. A collection of coalitions $\mathcal{A}$ is *totally mixed at* N if $\mathcal{A}=\cup_{i=1}^k\mathcal{P}^i$, where $\mathcal{P}^i$ are partitions of N and for each collection $\{S_i\}_{i=1}^k$ ($S_i\in\mathcal{P}^i$), we have $\cap_{i=1}^k S_i\neq\emptyset$.

Example 7. Let $N = \{1, 2, 3, 4\}$, $\mathcal{C} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2\}, \{3, 4\}\}$,
$\mathcal{B}^2 = \{\{1, 3\}, \{2, 4\}\}$,
then $\mathcal{C}$ is totally mixed at N.

Theorem 4. *Let $g \in \mathcal{G}$ and $g(t) \to +\infty$ as $t \to +\infty$. The g–solution of $(N, \mathcal{A}, c, v)$ coincides with the weakly proportional solution of $(N, \mathcal{A}, c, v)$ for all $c > 0$, v if and only if $\mathcal{A}$ is totally mixed at N.*

Proof. Let $\mathcal{A}$ be totally mixed at N. Then $\mathcal{A}$ is mixed at N and it follows from Theorem 3 that g–solution of $(N, \mathcal{A}, c, v)$ is always contained in the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$. Since $x(S)$ are uniquely defined for all $x \in \mathcal{WP}(N, \mathcal{A}, c, v)$, this implies coincidence of g–solution and the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$.

Now suppose that $\mathcal{WP}(N, \mathcal{A}, t, v)$ coincides with g–solution of $(N, \mathcal{A}, t, v)$ for all $c > 0$, all v with $v(T) > 0$. By Proposition 5, $\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{B}^i$, where $\mathcal{B}^i$ are subsets of partitions of N. If each $\mathcal{B}^i$ is a partition $\mathcal{P}^i$ of N then by Theorem 1, for each collection $\{S_i\}_{i=1}^{k}$ with $S_i \in \mathcal{P}^i$, we have $\cap_{i=1}^{k} S_i \neq \emptyset$, so $\mathcal{A}$ is totally mixed at N.

Let some $\mathcal{B}^i$ be not a partition of N. Then without loss of generality, there exists $q < k$ such that $\cup_{i=1}^{q} \mathcal{B}^i$ does not cover N and $\cup_{i=1}^{q} \mathcal{B}^i \cup \mathcal{B}^j$ covers N for each $j > q$. Denote $N^0 = \bigcup_{S \in \cup_{i=1}^{q} \mathcal{B}^i} S$. We consider 2 cases.

Case 1. For each $j = q + 1, \dots, k$, there exists $S_j \in \mathcal{B}^j$, such that if S_j is taken out from $\mathcal{B}^j$, then the remaining elements of $\cup_{j=q+1}^{k} \mathcal{B}^j$ cover $(N \setminus N^0)$.

Let $j_0 \in N \setminus N^0$, $\mathcal{A}_{j_0} = \{Q_i\}_{i \in M}$, then $Q_i \in \mathcal{B}^i$, $i \in \{q + 1, \dots, k\}$. Since $\mathcal{A}$ is mixed by Theorem 3, there exists $j_1 \in N$ such that $\mathcal{A}_{j_1} = \{S_i\}_{i \in M}$, then $j_1 \in N \setminus N^0$, hence Case 1 is impossible.

Case 2. If $S_j \in \mathcal{B}^j$ is taken out from $\mathcal{B}^j$, $j = q + 1, \dots, k$, then the remaining elements of $\cup_{j=q+1}^{k} \mathcal{B}^j$ do not cover $N \setminus N^0$.

For each $j = q + 1, \dots, k$, $S_j \in \mathcal{B}^j$, we have $S_j \cap (N \setminus N^0) \neq \emptyset$. Indeed, suppose that $S_{j_0} \subset N^0$ for some $j_0 > q$. Then if we take S_{j_0} and arbitrary $S_j \in \mathcal{B}^j$ for $j > q$, $j \neq j_0$ out from $\cup_{j=q+1}^{k} \mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^{k} \mathcal{B}^j$ cover $N \setminus N^0$ as if $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N.

Let

$$\mathcal{C} = \{(N \setminus N^0) \cap S : \; S \in \cup_{j=q+1}^{k} \mathcal{B}^j, \, S \cap P = \emptyset \text{ for some } P \in \mathcal{A}\}.$$

Note that $P, S \in \cup_{j=q+1}^{k} \mathcal{B}^j$, $P \neq S$, $P \cap (N \setminus N^0) \in \mathcal{C}$ imply $P \cap (N \setminus N^0) \neq S \cap (N \setminus N^0)$.

Indeed, suppose that $P \cap (N \setminus N^0) = S \cap (N \setminus N^0)$. There exists $P^1 \in \mathcal{A}$ such that $P \cap P^1 = \emptyset$. If we take S, P^1 and arbitrary $S_j \in \mathcal{B}^j$ for $j > q$ with $P \notin \mathcal{B}^j$ out from $\cup_{j=q+1}^{k} \mathcal{B}^j$, the remaining elements of $\cup_{j=q+1}^{k} \mathcal{B}^j$ cover $N \setminus N^0$ because $\{N^0\} \cup \mathcal{B}^{j_0}$ covers N, where $\mathcal{B}^{j_0} \ni S$, but this is impossible in the considered case.

For arbitrary problem $(N, \mathcal{A}, c, v)$, where $\mathcal{A}$ is under the case 2, consider the problem $(N \setminus N^0, \mathcal{C}, c, w)$, where $w(T) = v(S)$ for $T = S \cap (N \setminus N^0) \in \mathcal{C}$. As was proved above, w is well defined. Under the case 2, due to Theorem 1, there exists $y \in \mathcal{WP}(N \setminus N^0, \mathcal{C}, c, w)$. Let $x \in R^{|N|}$, $x_i = 0$ for $i \in N^0$, $x_i = y_i$ for $i \in N \setminus N^0$, then $x \in \mathcal{WP}(N, \mathcal{A}, c, v)$, $x(N^0) = 0$.

Let $\tilde{v}(S) = |S|/|N|$ for all $S \in \mathcal{A}$, $\tilde{x}_i = 1/|N|$ for all $i \in N$, then $\tilde{x}$ belongs to g–solution of $(N, \mathcal{A}, 1, \tilde{v})$ as if $\tilde{x}(S) = \tilde{v}(S)$ for all $S \in \mathcal{A}$, but $\tilde{x}(N^0) > 0$. Thus in case 2 g–solution does not coincide with the Weakly Proportional solution for some problem. □

Corollary 1. *The Proportional Nucleolus, g–solutions, and the Weakly Proportional solution of $(N, \mathcal{A}, c, v)$ coincide for all $c > 0$, all v with $v(T) > 0$ if and only if $\mathcal{A}$ is totally mixed at N.*

Definition 9. A collection of coalitions $\mathcal{A}$ is *strongly mixed at* N if $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, where
$\mathcal{B}^i$ is a partition of N for $i \le k_1$ where $0 \le k_1 \le k$;
$\mathcal{B}^i$ is a proper subset of a partition of N for $k_1 < i \le k$;
$Q \in \mathcal{B}^i$, $S \in \mathcal{B}^j$, and $i \ne j$ imply $Q \cap S \ne \emptyset$;
$|\mathcal{A}_i| = m$ for each $i \in N$, where $k_1 \le m \le k$;
for each $M_1 \subset \{k_1 + 1, \dots, k\}$ with $|M_1| = m - k_1$, $S_{j_t} \in \mathcal{B}^{j_t}$ for $t \in M = M_1 \cup \{1, \dots, k_1\}$ we have $\cap_{t \in M} S_{j_t} \ne \emptyset$.

Remark 3. If $\mathcal{A}$ is totally mixed at N then $\mathcal{A}$ is strongly mixed at N $(k = k_1)$.

Remark 4. If $\mathcal{A}$ is strongly mixed at N then $\mathcal{A}$ is mixed at N.

Remark 5. If $\mathcal{A}$ is mixed at N, $\mathcal{A} = \cup_{i=1}^{k} \mathcal{B}^i$, and $|\mathcal{A}_i| = k - 1$ for each $i \in N$, then $\mathcal{A}$ is strongly mixed at N.

Proof. For each $M = M_1 \cup \{1, \dots, k_1\}$ with $|M| = k-1$, if $q \notin M$, $q \in \{k_1+1, \dots, k\}$, then there exists $i \in N \setminus \bigcup_{T \in \mathcal{B}^q} T$. Since $|M| = k - 1$, $\mathcal{A}_i = M$. Then $\mathcal{A}$ is strongly mixed because $\mathcal{A}$ is mixed. □

Example 8. $N = \{1, 2, 3\}$, $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$, then all $\mathcal{B}^i$ are singletones and $\mathcal{A}$ is strongly mixed but not totally mixed.

Example 9. $N = \{1, 2, \dots, 12\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2 \cup \mathcal{B}^3$, where
$\mathcal{B}^1 = \{\{1, 2, 3, 4\}, \{5, 6, 7, 8\}\}$,
$\mathcal{B}^2 = \{\{3, 5, 9, 10\}, \{4, 6, , 11, 12\}\}$,
$\mathcal{B}^3 = \{\{1, 7, 9, 11\}, \{2, 8, 10, 12\}\}$.
Then $\mathcal{A}$ is strongly mixed at N but not totally mixed.

Example 10. Let $N = \{1, 2, \dots, 6\}$, $\mathcal{A} = \mathcal{B}^1 \cup \mathcal{B}^2$, where
$\mathcal{B}^1 = \{\{1, 2, 3\}, \{4, 5, 6\}\}$,
$\mathcal{B}^2 = \{\{1, 4\}, \{2, 5\}\}$,
then $\mathcal{A}$ is mixed at N but not strongly mixed.

Theorem 5. *Let $\mathcal{A}$ be strongly mixed at N. Then the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$ coincides with g–solution of $(N, \mathcal{A}, c, v)$ for all $g \in \mathcal{G}$, $c > 0$, v and is contained in the Weakly Proportional solution.*

Proof. Let $\mathcal{A}$ be strongly mixed. Then $\mathcal{A}$ is weakly mixed and, by Theoren 2, the Proportional Nucleolus is contained in the Weakly Proportional solution. We prove that the Proportional Nucleolus coincides with g–solution. If all $\mathcal{B}^i$ are partitions of

N, then g–solution coincides with the Weakly Proportional solution, hence it coincides with the Proportional Nucleolus. Let $k_i < k$. Let x belong to the Proportional Nucleolus of $(N, \mathcal{A}, c, v)$. Denote

$$\mathcal{A}_i^0 = \{T \in \mathcal{A}_i, T \notin \bigcup_{q=1}^{k_1} \mathcal{B}^q\}.$$

We prove that $x_i > 0$ implies

$$\max_{T \in \mathcal{A}_i^0 \setminus \mathcal{A}_j^0} x(T)/v(T) \leq \min_{S \in \mathcal{A}_j^0 \setminus \mathcal{A}_i^0} x(S)/v(S) \quad \text{for all} \quad j \in N \setminus \{i\}. \tag{11}$$

Suppose that (11) is not fulfilled, i.e., there exist $i_0, j_0 \in N$ and $Q, P \in \mathcal{A}$ such that $x_{i_0} > 0$, $Q \in \mathcal{A}_{i_0}^0 \setminus \mathcal{A}_{j_0}^0$, $P \in \mathcal{A}_{j_0}^0 \setminus \mathcal{A}_{i_0}^0$, and $x(Q)/v(Q) > x(P)/v(P)$. Since $\mathcal{A}$ is strongly mixed, there exists $j_1 \in \bigcap_{T \in \mathcal{A}_{i_0} \cap \{P\} \setminus \{Q\}} T$. Take $\delta > 0$ such that $\delta \leq x_{i_0}$ and $(x(Q) - \delta)/v(Q) > (x(P) + \delta)/v(P)$.

Let $y = \{y_i\}_{i \in N}$, $y_{i_0} = x_{i_0} - \delta$, $y_{j_1} = x_{j_1} + \delta$, $y_q = x_q$ otherwise. Then $x(P)/v(P) < y(P)/v(P) < y(Q)/v(Q) < x(Q)/v(Q)$ and $x(T) = y(T)$ for all $T \in \mathcal{A} \setminus \{P, Q\}$, but this contradicts the definition of the Proportional Nucleolus.

Since x is weakly proportional and g is increasing, $x_i > 0$ implies in view of (11)

$$\sum_{T \in \mathcal{A}_i \setminus \mathcal{A}_j} g(x(T)/v(T)) \leq \sum_{S \in \mathcal{A}_j \setminus \mathcal{A}_i} g(x(S)/v(S)) \quad \text{for all} \quad j \in N \setminus \{i\}.$$

Then, by Property 4, x belongs to g–solution of $(N, \mathcal{A}, c, v)$.

Since both g–solution and the Proportional Nucleolus are defined by $\{x(T)\}_{T \in \mathcal{A}}$, the Proportional Nucleolus and g–solution coincide. $\square$

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Principles of Stable Cooperation in Stochastic Games

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Abstract The paper considers stochastic games in the class of stationary strategies. The cooperative form of this class of stochastic games is constructed. The cooperative solution is found. Conditions of dynamic stability for stochastic games are obtained. Principles of dynamic stability include three conditions: subgame consistency, strategic stability and irrational behavior proof condition of the cooperative agreement. Also the paper considers the example for which the cooperative agreement is found and the conditions of dynamic stability are checked.

Keywords: cooperative stochastic game, stationary strategies, time consistency, subgame consistency, payoff distribution procedure, strategic stability, irrational behavior proof condition

1. Introduction

A stochastic game is a dynamic game process. If cooperation is possible in the game an important property of cooperative agreement is stability in dynamics. The work of Petrosyan and Zenkevich, 2009, considers three principles of stable cooperation: time-consistency (dynamic consistency), strategic stability and irrational behavior proof condition.

L.A. Petrosyan was the first to introduce the concept of dynamic consistency for differential games (Petrosyan, 1977). This condition appeared to be topical also for stochastic games (Petrosyan, 2006). The paper considers stochastic games in stationary strategies with the finite number of states any of which can be realized at every game stage. With this type of the definition of stochastic games the consistency of cooperative agreement should take place in every position (state) of the game. In other words, the claim of subgame consistency is laid to the cooperative agreement. Subgame consistency of the cooperative agreement lets the players expect receiving the allocation from the same optimality principle in every stochastic subgame.

The condition of strategic stability guarantees the existence of Nash equilibrium in the regularized game with the payoffs that players expect to receive as a result of the cooperative agreement. The regularization of the game is constructed on the basis of the initial stochastic game with the use of payoff distribution procedure (Petrosyan and Danilov, 1979). The conditions of strategic stability for stochastic games were also considered by Grauer and Petrosyan, 2002.

The irrational behavior proof condition (Yeung, 2006) lets guarantee that if some player (group of players) cancels the agreement at some stage of the game and players play individually from this stage to the end of the game they receive not less than if each player plays by himself during the whole game. This condition also lets secure the cooperative agreement from the force majeure circumstances.

The definition of stochastic games was introduced by Shapley, 1953a. At present a lot of papers deal with the study of stochastic games (Petrosyan et al., 2004, Petrosyan and Baranova, 2006, Herings and Peeters, 2004). Stochastic games have wide application in the field of telecommunication system modeling (Parilina, 2010, Altman et al., 2003), in economics (Amir, 2003), in the problem of tax evasion (Raghavan, 2006).

2. Stochastic games in stationary strategies

Stochastic game begins with the chance turn, i.e. with the choice of the initial state of the game which the game process begins with. The state of the stochastic game is determined as simultaneous normal form game of n players. One of the finite number of states is realized at each stage of the stochastic game. In the state some action profile is realized depending on which transition to the next states is accomplished with some probability. The payoff of the players is discounted when the game goes on. There are some notations:

- The set of players is $N = \{1, \dots, n\}$.
- The set of states is $\{\Gamma^j\}_{j=1}^t$ where $\Gamma^j = \langle N, X_1^j, \dots, X_n^j, K_1^j, \dots, K_n^j \rangle$ is state j, set N is equal for all Γ^j, $j = 1, \dots, t$, X_i^j is the finite set of pure strategies of player i in Γ^j, $K_i^j(x_1^j, \dots, x_n^j) = K_i^j(x^j)$ is a payoff function of player i in state Γ^j, $j = 1, \dots, t$.
- The probability that state Γ^k is realized if at the previous stage (in state Γ^j) action profile $x^j = (x_1^j, \dots, x_n^j)$ has realized, is $p(j, k; x^j)$. It is obvious that $p(j, k; x^j) \geqslant 0$ and $\sum_{k=1}^t p(j, k; x^j) = 1$ for each $x^j \in X^j = \prod_{i \in N} X_i^j$ and for any $j, k = 1, \dots, t$.
- The discount factor is $\delta \in (0, 1)$.
- The vector of the initial distribution on states $\Gamma^1, \dots, \Gamma^t$ is $\pi^0 = (\pi_1^0, \dots, \pi_t^0)$, where π_j^0 is the probability that state Γ^j is realized at the first stage of the game, $\sum_{j=1}^t \pi_j^0 = 1$;
- The set of player i's stationary strategies is $\Xi_i = \{\eta_i\}$. Using stationary strategies the player's choice of the strategy in each state from set $\left\{\Gamma^1, \dots, \Gamma^t\right\}$ at any stage depends only on which state is realized at this stage, i.e. $\eta_i : \Gamma^j \longmapsto x_i^j \in X_i^j, j = 1, \dots, t$.

Definition 1. Call the set

$$G = \left\langle N, \left\{\Gamma^j\right\}_{j=1}^t, \{\Xi_i\}_{i \in N}, \delta, \pi^0, \left\{p(j, k; x^j)\right\}_{j=\overline{1,t}, k=\overline{1,t}, x^j \in \prod\limits_{i=1}^n X_i^j} \right\rangle \tag{1}$$

finite stochastic game in stationary strategies.

Definition 2. Call stochastic game (12) with vector $\pi^0 = (0, \dots, 0, 1, 0, \dots, 0)$ (with 1 in the jth component), i.e. the game beginning with state Γ^j, the finite stochastic subgame in stationary strategies and denote as G^j, $j = 1, \dots, t$.

Remark 1. Obviously, player i's stationary strategy in game G is player i's stationary strategy in any subgame $G^1, \dots, G^t$.

Payoff in finite stochastic game is a random variable. So we have to determine the utility function of the payoff. Consider the mathematical expectation of the player's payoff as the utility of his payoff in stochastic game G. Let $\overline{E}_i(\eta)$ be the expected payoff of player i in game G and $E_i^j(\eta)$ be the expected payoff of player i in subgame G^j when strategy profile η is realized in stochastic game G (subgame G^j). Form vector $E_i(\eta) = (E_i^1(\eta), \dots, E_i^t(\eta))$.

For the expected payoff of player i in subgame G^j the following recurrent equation takes place:

$$E_i^j(\eta) = K_i^j(x^j) + \delta \sum_{k=1}^{t} p(j,k;x^j) E_i^k(\eta) \tag{2}$$

under condition that $\eta(\Gamma^j) = x^j$, i.e. $\eta(\cdot) = (\eta_1(\cdot), \dots, \eta_n(\cdot))$ where $\eta_i(\Gamma^j) = x_i^j \in X_i^j$, $x^j = (x_1^j, \dots, x_n^j)$ for each $j = 1, \dots, t$, $i \in N$.

Since stochastic game G is considered in the class of stationary strategies defined above and the set of states $\{\Gamma^1, \dots, \Gamma^t\}$ is finite then it is sufficiently to consider t of subgames $G^1, \dots, G^t$ accordingly beginning with the states $\Gamma^1, \dots, \Gamma^t$.

Hereinafter, let $\eta(\cdot) = (\eta_1(\cdot), \dots, \eta_n(\cdot))$ be the stationary strategy profile such as $\eta_i(\Gamma^j) = x_i^j \in X_i^j$ where $j = 1, \dots, t$, $i \in N$. We restrict our consideration to the set of player i's pure stationary strategies in stochastic game G. Denote it as $\widetilde{\Xi}_i$.

The matrix of transition probabilities in stochastic game G under the realization of stationary strategy profile $\eta(\cdot)$ looks like:

$$\Pi(\eta) = \begin{pmatrix} p(1,1;x^1) & \dots & p(1,t;x^1) \\ p(2,1;x^2) & \dots & p(2,t;x^2) \\ \dots & \dots & \dots \\ p(t,1;x^t) & \dots & p(t,t;x^t) \end{pmatrix}. \tag{3}$$

We can rewrite equation (2) in matrix form using (3) as follows:

$$E_i(\eta) = K_i(\eta) + \delta \Pi(\eta) E_i(\eta), \tag{4}$$

where $K_i(\eta) = (K_i^1(x^1), \dots, K_i^t(x^t))$, and $K_i^j(x^j)$ is the player i's payoff in state Γ^j on condition that strategy profile $x^j \in X^j$ is realized in this state.

Equation (3) is equivalent to the following one:

$$E_i(\eta) = \left(\mathbb{I} - \delta \Pi(\eta)\right)^{-1} K_i(\eta), \tag{5}$$

where $\mathbb{I}$ is an identity $t \times t$ matrix.

Remark 2. Matrix $\left(\mathbb{I} - \delta \Pi(\eta)\right)^{-1}$ always exists for $\delta \in (0,1)$. It is not difficult to prove this statement. It is known that all the eigenvalues of stochastic matrix $\Pi(\eta)$ are in the interval $[-1,1]$. For the existence of matrix $\left(\mathbb{I} - \delta \Pi(\eta)\right)^{-1}$ it is necessary and sufficient that the determinant of matrix $(\Pi(\eta) - \frac{1}{\delta}\mathbb{I})$ be not equal to zero. So matrix $(\Pi(\eta) - \frac{1}{\delta}\mathbb{I})$ must not have the eigenvalue equal to $\frac{1}{\delta}$. The last condition takes place because $\frac{1}{\delta} > 1$, so this number cannot be the eigenvalue of stochastic matrix $\Pi(\eta)$.

The expected payoff of player i in the stochastic game G we can find in the following way:

$$\overline{E}_i(\eta) = \pi^0 E_i(\eta). \tag{6}$$

3. Cooperation in stochastic games

Suppose now, that the players from N decide to cooperate to receive the maximum total payoff. Denote the pure strategy profile maximizing the sum of the expected players' payoffs in stochastic game G as $\overline{\eta}(\cdot) = (\overline{\eta}_1(\cdot), \ldots, \overline{\eta}_n(\cdot))$, i.e.

$$\max_{\eta \in \prod\limits_{i \in N} \widetilde{\Xi}_i} \sum_{i \in N} \overline{E}_i(\eta) = \sum_{i \in N} \overline{E}_i(\overline{\eta}). \tag{7}$$

Problem (6) may have more than one decision. Call strategy profile $\overline{\eta}(\cdot)$ satisfying (6) as cooperative decision.

The coalitional form of noncooperative game is usually given by the pair $\langle N, V\rangle$, where N is the set of players and V is a real-valued function, called the characteristic function of the game, defined on the set 2^N (the set of all subsets of N), and satisfying two properties (1) $V(\emptyset) = 0$, and (2) (superadditivity) for any disjoint coalitions $S, T \subset N$, $S \cap T = \emptyset$, the next inequality is satisfied: $V(S) + V(T) \leqslant V(S \cup T)$. The value $V(S)$ is a real number for each coalition $S \subset N$, which may be interpreted as the worth or power of coalition S when its members play together as a unit. Condition (2) says that the value of two disjoint coalitions is at least as great when they play together as when they work apart. The assumption of superadditivity is not needed for some of the theory of coalitional games, but it seems to be a natural condition.

Define the characteristic function $\overline{V}(S)$ in stochastic game G via characteristic function $V^j(S)$ of stochastic subgames G^j, $j = 1, \ldots, t$, as follows:

$$\overline{V}(S) = \pi^0 V(S) \tag{8}$$

for any coalition $S \subset N$ where $V(S) = (V^1(S), \ldots, V^t(S))$, $V^j(S)$ is the value of the characteristic function of stochastic subgame G^j derived for coalition S.

The task is to determine the characteristic function $V^j(S)$ for any coalition S.

Firstly, consider $S = N$. Bellman equation (Bellman, 1957) for the value $V(N)$ can be written as follows:

$$V(N) = \max_{\eta \in \prod\limits_{i \in N} \widetilde{\Xi}_i} \left[\sum_{i \in N} K_i(\eta) + \delta \Pi(\eta) V(N) \right] = \sum_{i \in N} K_i(\overline{\eta}) + \delta \Pi(\overline{\eta}) V(N),$$

where $\overline{\eta}(\cdot)$ is the pure strategy profile satisfying condition (6).

The value $V(N)$ is got from the previous equation:

$$V(N) = (\mathbb{I} - \delta \Pi(\overline{\eta}))^{-1} \sum_{i \in N} K_i(\overline{\eta}). \tag{9}$$

Secondly, consider $S \subset N$, $S \neq \emptyset$. To define the value of characteristic function $V^j(S)$ for this coalition, $j = 1,\ldots,t$, for each subgame G^j, define an auxiliary zero-sum stochastic game G^j_S where coalition $S \subset N$ plays as a maximizing player and coalition $N \backslash S$ plays as a minimizing player. Define the value of function $V^j(S)$ for subgame G^j as a lower value of antagonistic stochastic game G^j_S in pure stationary strategies (in fact, the lower value of matrix game):

$$V^j(S) = \max_{\eta_S} \min_{\eta_{N \backslash S}} \sum_{i \in S} E^j_i(\eta_S, \eta_{N \backslash S}), \tag{10}$$

where $(\eta_S(\cdot), \eta_{N\setminus S}(\cdot))$ is a strategy profile in pure stationary strategies and $\eta_S(\cdot) = (\eta_{i_1}(\cdot), \ldots, \eta_{i_k}(\cdot))$ is a vector of stationary strategies of players $i_1, \ldots, i_k \in S$, $i_1 \bigcup \ldots \bigcup i_k = S$, $\eta_S(\cdot) \in \prod_{j=1}^{k} \widetilde{\Xi}_{i_j}$, the set of pure stationary strategies of coalition $S \subset N$, and $\eta_{N\setminus S}(\cdot)$ is a vector of stationary strategies of players $i_{k+1}, \ldots, i_n \in N\setminus S$, $i_{k+1} \bigcup \ldots \bigcup i_n = N\setminus S$, $\prod_{j=k+1}^{n} \widetilde{\Xi}_{i_j}$, the set of pure stationary strategies of coalition $N\setminus S$.

Finally, consider $S = \emptyset$ and get the value of characteristic function:

$$V^j(\emptyset) = 0. \tag{11}$$

Remark 3. Characteristic functions $\overline{V}(S)$ determined by (10) and $V^j(S)$ determined by (7)—(11) are superadditive.

Definition 3. Cooperative stochastic subgame G^j_{co} is a set $\langle N, V^j(\cdot)\rangle$, where N is the set of players, and $V^j : 2^N \longrightarrow R$ is the characteristic function calculated by (7) – (11).

Definition 4. Cooperative stochastic game G_{co} is a set $\langle N, \overline{V}(\cdot)\rangle$, where N is the set of players and $\overline{V} : 2^N \longrightarrow R$ is the characteristic function calculated by (10).

Definition 5. Vector $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_n)$ satisfying the two following conditions:
1) $\sum_{i\in N} \alpha^j_i = V^j(N)$,
2) $\alpha^j_i \geqslant V^j(\{i\})$ for any $i \in N$,
is called the allocation in subgame G^j_{co} $(j = 1, \ldots, t)$. Denote the set of allocations in cooperative subgame G^j_{co} as A^j, $j = 1, \ldots, t$.

Definition 6. Vector $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n)$, where $\overline{\alpha}_i = \pi^0\alpha_i$, $\alpha_i = (\alpha^1_i, \ldots, \alpha^t_i)$, and $(\alpha^j_1, \ldots, \alpha^j_n) = \alpha^j \in I^j$ is called an allocation in cooperative stochastic game G_{co}. Denote the set of allocations in cooperative stochastic game G_{co} as $\overline{I}$.

Suppose that the set of allocations in any subgame G^j_{co}, $j = 1, \ldots, t$, is nonempty. So the set of allocations in cooperative stochastic game G_{co} is also nonempty.

4. Principles of stable cooperation

4.1. Subgame consistency of cooperative agreement

Suppose that players cooperate in stochastic game and for every subgame G^j_{co} choose allocation $\alpha^j = (\alpha^j_1, \ldots, \alpha^j_n) \in I^j$. The problem is how to realize payments to the players at each stage of the stochastic game for getting the expected payoff α^j_i for player i in stochastic subgame G^j. If players receive payoffs according to their payoff functions in the states they hardly ever get the components of the chosen allocation in mathematical expectation sense. To find the way out of the situation we should suggest the method of redistribution of total players' payoff in every state realized in the stochastic game process. This method was proposed by Petrosyan and Danilov, 1979, for differential games.

There are two principles of constructing the real payments to the players in the dynamic game adapted to the theory of stochastic games:

1. The sum of payments to the players in every state is equal to the sum of players' payoffs in strategy profile realized in this state according to cooperative decision $\overline{\eta}(\cdot)$.
2. The expected sum of payments to player i in each subgame G^j is equal to the ith component of an allocation in subgame G^j_{co} that players have chosen before the beginning of the game.

Taking into account that in stochastic game (12) the number of subgames is equal to the number of possible states we should find vector $\beta_i = (\beta_i^1, \ldots, \beta_i^t)$ for every $i \in N$, where β_i^j is a payment to player i in state Γ^j, $j = 1, \ldots, t$. And these payments have to satisfy the two above principles and if so then these payments can be called payoff distribution procedure (PDP) (see Petrosyan and Danilov, 1979).

Find the conditions that the new payoffs of the players are satisfied the principles of PDP in the terms of stochastic games.

1. The first principle is equivalent to the following equation:

$$\sum_{i \in N} \beta_i^j = \sum_{i \in N} K_i^j(\overline{x}^j), \tag{12}$$

   where $\overline{x}^j$ is an action profile realized under cooperative decision $\overline{\eta}(\cdot)$ in state Γ^j, $j = 1, \ldots, t$.
2. To find the condition of the second principle we need to work out the expected total payoff of player i in stochastic subgame with new payments β_i^j in state Γ^j, $j = 1, \ldots, t$. Denote this value as B_i and write the recurrent equation for this value:

$$B_i^j = \beta_i^j + \delta \sum_{k=1}^{t} p(j, k; x^j) B_i^k,$$

   or in vector form:

$$B_i = \beta_i + \delta \Pi(\overline{\eta}) B_i, \tag{13}$$

   where $B_i = (B_i^1, \ldots, B_i^t)$. Equation (13) is equivalent to the following one:

$$B_i = \left(\mathbb{I} - \delta \Pi(\eta)\right)^{-1} \beta_i. \tag{14}$$

   In respect to the second principle of PDP and equation (14) we obtain the following equation for β_i:

$$\alpha_i = \left(\mathbb{I} - \delta \Pi(\eta)\right)^{-1} \beta_i, \tag{15}$$

   where $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^t)$, $(\alpha_1^j, \ldots, \alpha_n^j) = \alpha^j \in I^j$. Equation (15) can be rewritten in an equivalent form:

$$\beta_i = (\mathbb{I} - \delta \Pi(\overline{\eta})) \alpha_i. \tag{16}$$

   It is easy to show that β_i found from (16) satisfies (12). As $\sum_{i \in N} \beta_i^j$ is equal to $(\mathbb{I} - \delta \Pi(\overline{\eta})) \sum_{i \in N} \alpha_i = (\mathbb{I} - \delta \Pi(\overline{\eta})) V(N)$, and $V(N)$ is found from (7) then equation (12) holds.

Remark 4. Equation (16) equals the following functional equation

$$\alpha_i = \beta_i + \delta \varPi(\overline{\eta})\alpha_i. \tag{17}$$

The second item in the right part of equation (17) is the expected value of the component of the allocation in subgame beginning with the next stage. Suppose that the allocation for each subgame is chosen from the same optimality principle that has been chosen by the players at the beginning of the game.

Obviously, if players keep to cooperative decision $\overline{\eta}(\cdot)$ the expected payoff of player i in stochastic game with new payments in some action profiles (which are realized in the states under cooperative decision $\overline{\eta}$ is equal to the expected value of the correspondent component of the allocation in cooperative stochastic game G_{co}.

Now for every allocation $\overline{\alpha} = (\overline{\alpha}_1, \ldots, \overline{\alpha}_n)$, where $\overline{\alpha}_i = \pi^0\alpha_i$, $\alpha_i = (\alpha_i^1, \ldots, \alpha_i^t)$, $(\alpha_1^j, \ldots, \alpha_n^j) = \alpha^j \in I^j$ we can determine the regularization of stochastic game G by the following definition.

Definition 7. Noncooperative stochastic game G_α (subgame G_α^j, $j = 1, \ldots, t$) is called α-regularization of stochastic game G (subgame G^j), if for any player $i \in N$ in state $\varGamma^j$ payoff function $K_i^{\alpha,j}(x^j)$ is defined as follows:

$$K_i^{\alpha,j}(x^j) = \begin{cases} \beta_i^j, & \text{if } x^j = \overline{x}^j; \\ K_i^j(x^j), & \text{if } x^j \neq \overline{x}^j, \end{cases} \tag{18}$$

where PDP $\beta = (\beta_1, \ldots, \beta_n)$ (Petrosyan and Baranova, 2006) is found from (**??**).

The procedure of regularization of the stochastic game G (subgame G^j) suggests a method of construction of real payments to the players in every state and one can insist that players are interested in the redistribution of their payoffs because getting $\beta_i^1, \ldots, \beta_i^t$ in states $\varGamma^1, \ldots, \varGamma^t$ respectively player i receives the same sum (in terms of mathematical expectation) in game G_α (G_α^j) as he has planned to receive in the cooperative stochastic game G_{co} (G_{co}^j) and the expected sum of the remained payments will belong to the same optimality principle which has been chosen by the players at the beginning of the game. In this case we can say that *subgame consistency* (*dynamic consistency*) of the chosen cooperative agreement takes place.

4.2. Strategic stability of cooperative agreement

Introduce the additional notations. Set $\varGamma(k)$ as the state realized at stage k of stochastic game G. It is obvious that $\varGamma(k) \in \{\varGamma^1, \ldots, \varGamma^t\}$. Write $x(k)$ the strategy profile realized in state $\varGamma(k)$. Set the subgame of stochastic game G_α from Definition 3 beginning from state $\varGamma(k)$ as $G_\alpha^{\varGamma(k)}$.

Call the sequence $((\varGamma(1), x(1)), (\varGamma(2), x(2)), \ldots, (\varGamma(k-1), x(k-1)))$ the history of stage k and denote it as $h(k)$. Let T be $\{(\varGamma^1, \overline{x}^1), (\varGamma^2, \overline{x}^2), \ldots, (\varGamma^t, \overline{x}^t)\}$.

Stochastic game G and G_α are games with perfect information in the sense that at each stage k ($k = 1, 2, \ldots$) all players from N know state $\varGamma(k)$ and the history of stage k.

Definition 8. We call behavior strategy profile $\varphi^*(\cdot) = (\varphi_1^*(\cdot),\ldots,\varphi_n^*(\cdot))$ strong transferable equilibrium in regularized game G_α if for any coalition $S \subset N$, $S \neq \emptyset$, the inequality

$$\sum_{i \in S} \overline{E}_i^\alpha(\varphi^*) \geqslant \sum_{i \in S} \overline{E}_i^\alpha(\varphi^* \parallel \varphi_S) \tag{19}$$

is true for any behavior strategy of coalition S: $\varphi_S(\cdot) = \{\varphi_i(\cdot)\}_{i \in S} \in \prod_{i \in S} \Phi_i$, and $\overline{E}_i^\alpha(\cdot)$ is the expected payoff of player i in α-regularization of stochastic game G.

Theorem 1. *If in α-regularization of stochastic game G with α such that $\overline{\alpha} = \pi^0\alpha$, the following inequality holds for any coalition $S \subset N$, $S \neq \emptyset$:*

$$\sum_{i \in S} \beta_i \geqslant (\mathbb{I} - \delta \Pi(\overline{\eta}))F(S), \tag{20}$$

where $F(S) = (F^1(S), \ldots, F^t(S))$,
$F^j(S) = \max\limits_{\substack{x_S^j \in \prod_{i \in S} X_i^j \\ x_S^j \neq \overline{x}_S^j}} \left\{ \sum\limits_{i \in S} K_i^j(\overline{x}^j \parallel x_S^j) + \delta \sum\limits_{l=1}^t p(j,l;\overline{x}^j \parallel x_S^j) V^l(S) \right\}$, *then in regularized game G_α there exists a strong transferable equilibrium with payoffs $(\overline{\alpha}_1,\ldots,\overline{\alpha}_n)$.*

Proof. Consider the behavior strategy profile $\widehat{\varphi}(\cdot) = (\widehat{\varphi}_1(\cdot), \ldots, \widehat{\varphi}_n(\cdot))$ in game G_α:

$$\widehat{\varphi}_i(h(k)) = \begin{cases} \overline{x}_i^j, & \text{if } \Gamma(k) = \Gamma^j, j = \overline{1,t}, h(k) \subset T; \\ \widehat{x}_i^j(S), & \text{if } \Gamma(k) = \Gamma^j,\ j = \overline{1,t},\ \exists\ l \in [1, k-1] \\ & \text{and } S \subset N,\ i \notin S\text{: } h(l) \subset T, \\ & \text{and } (\Gamma(l), x(l)) \notin T, \\ & \text{but } (\Gamma(l), (x(l) \parallel \overline{x}_S(l)) \in T, \\ \text{anyone} & \text{in other cases,} \end{cases} \tag{21}$$

where $\widehat{x}_i^j(S)$ is the player i's pure strategy in state Γ^j which with strategies $x_p^j(S)$, $p \neq i$, $p \notin S$ forms the strategy of coalition $\{N \backslash S\}$ in antagonistic game against coalition S in subgame $G^{\Gamma(j)}$.

The proof of the theorem repeats the proof of folk theorems (see Dutta, 1995) using the structure of strategy (21). Prove that $\widehat{\varphi}(\cdot) = (\widehat{\varphi}_1(\cdot), \ldots, \widehat{\varphi}_n(\cdot))$ determined in (21) is a strong transferable equilibrium in stochastic game G_α.

From definition (21) it follows that on condition that all players keep cooperative decision $\overline{\eta}(\cdot)$ the expected payoff of coalition S in subgame G_α^j, $j = 1, \ldots, t$, is equal to the following one:

$$E_S^j(\widehat{\varphi}(\cdot)) = \sum_{i \in S} E_i^j(\widehat{\varphi}(\cdot)) = \sum_{i \in S} E_i^j(\overline{\eta}(\cdot)).$$

Let $E_S(\widehat{\varphi}(\cdot))$ be equal to vector $(E_S^1(\widehat{\varphi}(\cdot)), \ldots, E_S^t(\widehat{\varphi}(\cdot)))$ then for any coalition $S \subset N$, $S \neq \emptyset$ the next equality takes place:

$$E_S(\widehat{\varphi}) = (\mathbb{I} - \delta \Pi(\overline{\eta}))^{-1} \sum_{i \in S} \beta_i. \tag{22}$$

Consider strategy-profile $(\widehat{\varphi}(\cdot) \parallel \varphi_S(\cdot))$, $S \subset N$, $S \neq \emptyset$, when some coalition S deviates from strategy $\widehat{\varphi}_S(\cdot)$. Let stage k be such that there exists number $l \in [1, k-1]$ such that history $h(l) \subset T$ and state $(\Gamma(l), x(l)) \notin T$ but $(\Gamma(l), (x(l) \parallel \overline{x}_S(l))) \in T$. Without loss of generality suggest that $\Gamma(k) = \Gamma^j$. Determine the payoff of coalition S in game G_α in strategy profile $(\widehat{\varphi}(\cdot) \parallel \varphi_S(\cdot))$ by formula $\sum\limits_{i \in S} \overline{E}_i^\alpha(\widehat{\varphi} \parallel \varphi_S) = \pi^0 \sum\limits_{i \in S} E_i^\alpha(\widehat{\varphi} \parallel \varphi_S)$, where

$$\sum_{i \in S} E_i^\alpha(\widehat{\varphi} \parallel \varphi_S) = \sum_{i \in S} E_i^{\alpha,[1,k-1]}(\widehat{\varphi} \parallel \varphi_S) + \delta^{k-1} \Pi^{k-1}(\widehat{\varphi} \parallel \varphi_S) \sum_{i \in S} E_i^{\alpha,[k,\infty)}(\widehat{\varphi} \parallel \varphi_S), \quad (23)$$

where the first term in the right side of equation (23) is the expected payoff of coalition S at the first $k-1$ stages of game G_α, $\sum\limits_{i \in S} E_i^{\alpha,[k,\infty)}(\widehat{\varphi} \parallel \varphi_S)$ in the second term is the expected payoff of coalition S in the subgame of game G_α beginning from stage k. Since there were no deviations of any coalition from the cooperative decision $\overline{\eta}(\cdot)$ up to stage $k-1$ inclusive as it was shown before the following equalities holds for the elements of the right side of (23):

$$\sum_{i \in S} E_i^{\alpha,[1,k-1]}(\widehat{\varphi} \parallel \varphi_S) = \sum_{i \in S} E_i^{\alpha,[1,k-1]}(\overline{\eta}),$$

$$\Pi^{k-1}(\widehat{\varphi} \parallel \varphi_S) = \Pi^{k-1}(\overline{\eta}).$$

In the second term of the right side of (23) as $E_i^{\alpha,[k,\infty)}(\widehat{\varphi} \parallel \varphi_S)$ we mean vector $(E_i^{\alpha,1}(\widehat{\varphi} \parallel \varphi_S), \ldots, E_i^{\alpha,t}(\widehat{\varphi} \parallel \varphi_S))$ where $E_i^{\alpha,j}(\widehat{\varphi} \parallel \varphi_S)$ is the expected payoff of player $i \in S$ in regularized subgame G_α^j beginning with state Γ^j.

Find the expected payoff of coalition S in subgame G_α^j beginning with stage k and state $\Gamma(k)$ is equal to Γ^j. The following formula takes place:

$$\sum_{i \in S} E_i^{\alpha,j}(\widehat{\varphi} \parallel \varphi_S) = \sum_{i \in S} K_i^j(\overline{x}^j \parallel x_S^j) + \delta \sum_{l=1}^{t} p(j,l;\overline{x}^j \parallel x_S^j) V^l(S), \quad (24)$$

because players from coalition $N \setminus S$ will punish coalition S playing the antagonistic game against coalition S beginning from stage $k+1$ according to the definition of strategy profile $\widehat{\varphi}(\cdot)$. In (24) the value of characteristic function $V^j(S)$ is determined by (9).

Since the expected payoffs of coalition S in strategy profiles $\widehat{\varphi}(\cdot)$ and $(\widehat{\varphi}(\cdot) \parallel \varphi_S(\cdot))$ are equal up to stage $k-1$, then as a result of deviation coalition S can guarantee the increase of payoff only at the expense of the part of game G_α beginning with stage k, i.e. at the expense of the expected payoff in subgame G_α^j, $j = 1, \ldots, t$. Coalition S in strategy profile $(\widehat{\varphi}(\cdot) \parallel \varphi_S(\cdot))$ can guarantee the following expected payoff from stage k:

$$\max_{\substack{x_S^j \in \prod_{i \in S} X_i^j \\ x_S^j \neq \overline{x}_S^j}} \left\{ \sum_{i \in S} K_i^j(\overline{x}^j \parallel x_S^j) + \delta \sum_{l=1}^{t} p(j,l;\overline{x}^j \parallel x_S^j) V^l(S) \right\}. \quad (25)$$

According to the definition of PDP the expected payoff of coalition S in regularized subgame G^j_α in strategy profile $\widehat{\varphi}(\cdot)$ can be found from the equation:

$$\sum_{i\in S} E^\alpha_i(\widehat{\varphi}) = (\mathbb{I} - \delta\Pi(\overline{\eta}))^{-1}\sum_{i\in S}\beta_i, \tag{26}$$

where $E^\alpha_i(\widehat{\varphi}(\cdot)) = (E^{\alpha,1}_i(\widehat{\varphi}(\cdot)), \ldots, E^{\alpha,t}_i(\widehat{\varphi}(\cdot))$. Taking into account inequality (20) from (25), (26) and reasoning presented above we can obtain inequality

$$E^\alpha_S(\widehat{\varphi}(\cdot)) \geqslant E^\alpha_S(\widehat{\varphi}(\cdot) \parallel \varphi_S(\cdot)).$$

Thus behavior strategy profile (21) is the strong transferable equilibrium in α-regularization of game G. The expected payoff of player i in game G_α in strategy-profile $\widehat{\varphi}(\cdot)$ is equal to $\overline{\alpha}_i$ where $\overline{\alpha}_i = \pi^0\alpha_i$ and vector $\alpha_i = (\alpha^1_i, \ldots, \alpha^t_i)$ consists of ith components of allocations α^1, …, α^t derived for cooperative subgames G^1, …, G^t accordingly.

Corollary 1. *If α-regularization of game G for any player $i \in N$ the next inequality takes place:*

$$\beta_i \geqslant (\mathbb{I} - \delta\Pi(\overline{\eta}))W_i,$$

where $W_i = (W^1_i, \ldots, W^t_i)$,
$W^j_i = \max\limits_{\substack{x^j_i\in X^j_i\\ x^j_i\neq\overline{x}^j_i}}\left\{K^j_i(\overline{x}^j \parallel x^j_i) + \delta\sum\limits_{l=1}^{t} p(j,l;\overline{x}^j \parallel x^j_i)V^l(\{i\})\right\}$, *then in α-regularization of stochastic game G there exists Nash equilibrium with players' payoffs* $(\overline{\alpha}_1, \ldots, \overline{\alpha}_n)$.

4.3. Condition of irrational behavior proofness

To protect the players against losses in cases when cooperation breaks up at some stage of the game it is necessary that the following equation takes place for every $i \in N$ and any $k = 1, 2, \ldots$.

$$V(\{i\}) \leqslant E^{\alpha,[1,k]}_i + \delta^k\Pi^k(\overline{\eta})V(\{i\}), \tag{27}$$

where $E^{\alpha,[1,k]}_i$ is the mathematical expectation of player i's payoff at the first k stages of regularized game G_α.

We suppose that before the beginning of the next game stage players know if the cooperation has broken up or not, i.e. information delay is not supposed in such a problem definition. In the left side of inequality (27) there is the value of characteristic function $V(\{i\}) = (V^1(\{i\}), \ldots, V^t(\{i\}))$ derived for player i where $V^j(\{i\})$ is the value of characteristic function of player i in subgame G^j. In the right side of inequality (27) the first term is equal to the expected value of player i's payoff if at the first k stages of the game players keep to cooperative decision $\overline{\eta}(\cdot)$, the second term is the expected payoff of player i beginning with stage $k+1$ if player i plays independently from this stage.

Proposition 1. *In stochastic game G_α for condition of irrational behavior proofness it is sufficient that the following inequality takes place for any $i \in N$:*

$$(\mathbb{I} - \delta\Pi(\overline{\eta}))(\alpha_i - V(\{i\})) \geqslant 0, \tag{28}$$

where $\alpha_i = (\alpha^1_i, \ldots, \alpha^t_i)$ and α^j_i is the ith component of allocation $\alpha^j \in I^j$.

Proof. Show condition (28) is sufficient for inequality (27) for any $k = 1, 2, \ldots$. We use the mathematical induction method for the proof.

Rewrite inequality (27) for $k = 1$:

$$V(\{i\}) \leqslant \beta_i + \delta \Pi(\overline{\eta}) V(\{i\}). \tag{29}$$

Transform inequality (28) considering definition α_i using PDP (15) and get inequality (29).

Suppose that from the truth of inequality (28) the truth of inequality (27) for $k = l$ follows. Rewrite inequality (27) for $k = l$:

$$V(\{i\}) \leqslant \beta_i + \ldots + \delta^{l-1} \Pi^{l-1}(\overline{\eta})\beta_i + \delta^l \Pi^l(\overline{\eta}) V(\{i\}). \tag{30}$$

Proof the statement for $k = l + 1$. Inequality (27) for $k = l + 1$ is as follows:

$$V(\{i\}) \leqslant \beta_i + \ldots + \delta^l \Pi^l(\overline{\eta})\beta_i + \delta^{l+1} \Pi^{l+1}(\overline{\eta}) V(\{i\}). \tag{31}$$

We should proof that if (28) is true then inequality (27) takes place for $k = l + 1$. After the transformation the right part of inequality (31) will have the following form:

$$\beta_i + \delta \Pi(\overline{\eta}) \left\{ \beta_i + \delta \Pi(\overline{\eta})\beta_i + \ldots + \delta^{l-1} \Pi^{l-1}(\overline{\eta})\beta_i + \delta^l \Pi^l(\overline{\eta}) V(\{i\}) \right\}$$

Taking into account inequality (30) the expression in braces is not less than $V(\{i\})$, so the right part of inequality (31) is not less than $\beta_i + \delta \Pi(\overline{\eta}) V(\{i\})$. Considering the definition of PDP (16) and inequality (28) we get the truth of inequality (27) for $k = l + 1$. So the statement is proved.

5. Example

Consider the following stochastic game G:

1. The set of players is $N = \{1, 2\}$.
2. The set of states is $\{\Gamma^1, \Gamma^2\}$, where $\Gamma^j = \langle N, X_1^j, X_2^j, K_1^j, K_2^j \rangle$, $j = 1, 2$, $X_1^j = \{x_{11}^j, x_{12}^j\}$ is the set of actions of player 1, and $X_2^j = \{x_{21}^j, x_{22}^j\}$ is the set of actions of player 2. For state Γ^1 players' payoffs are determined as follows:
$$\begin{pmatrix} (2;2) & (1;12) \\ (11;3) & (5;4) \end{pmatrix}.$$
And for state Γ^2 players' payoffs are determined as follows:
$$\begin{pmatrix} (3;1) & (2;7) \\ (8;2) & (5;3) \end{pmatrix}.$$
3. Transition probabilities from state Γ^1 look like
$$\begin{pmatrix} (0.6;0.4) & (0.7;0.3) \\ (0.3;0.7) & (0.6;0.4) \end{pmatrix},$$
where element (k, l) of the matrix consists of the transition probability from state Γ^1 to states Γ^1, Γ^2 accordingly on condition that player 1 chooses kth action and player 2 chooses lth action in state Γ^1.
Transition probabilities from state Γ^2 look like
$$\begin{pmatrix} (0.8;0.2) & (0.3;0.7) \\ (0.3;0.7) & (0.2;0.8) \end{pmatrix}.$$

4. The discount factor is $\delta = 0.99$.
5. The vector of the initial distribution on the set of states is $\pi^0 = (1/2, 1/2)$.

Determine the cooperative form G_{co} of game G described above. Firstly, calculate the cooperative decision $\overline{\eta} = (\overline{\eta}_1, \overline{\eta}_2)$ in stationary strategies using (5) and (6). We obtain the unique stationary strategy profile $\overline{\eta}_1(\Gamma^1) = x^1_{11}$, $\overline{\eta}_1(\Gamma^2) = x^2_{12}$, $\overline{\eta}_2(\Gamma^1) = x^1_{22}$, $\overline{\eta}_2(\Gamma^2) = x^2_{21}$.

Secondly, work out the values of characteristic function $V(\cdot) = (V^1(\cdot), V^2(\cdot))$ for all possible coalitions using (7)-(11):

$$V(\emptyset) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \ V(\{1\}) = \begin{pmatrix} 500.00 \\ 500.00 \end{pmatrix}, \ V(\{2\}) = \begin{pmatrix} 334.44 \\ 332.78 \end{pmatrix}, \ V(\{1,2\}) = \begin{pmatrix} 1152.48 \\ 1147.52 \end{pmatrix}.$$

Using (10) calculate the values of characteristic function $\overline{V}(\cdot)$ for all possible coalitions:

$$\overline{V}(\emptyset) = 0.00, \ \overline{V}(\{1\}) = 500.00, \ \overline{V}(\{2\}) = 333.61, \ \overline{V}(\{1,2\}) = 1150.00.$$

So, we determine the cooperative stochastic subgame G^j_{co} as the set $\langle N, V^j(\cdot)\rangle$, $j = 1, 2$, and cooperative stochastic game G_{co} as the set $\langle N, \overline{V}(\cdot)\rangle$.

Finally, suppose that players choose for example Shapley value (Shapley, 1953b) as allocation of their total payoff in cooperative stochastic game G_{co} and in all subgames G^j_{co}, $j = 1, 2$.

Shapley values calculated for subgames look like:

$$\alpha_1 = \begin{pmatrix} 659.02 \\ 657.37 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 493.46 \\ 490.15 \end{pmatrix},$$

where $\alpha_i = (\alpha^1_i, \alpha^2_i)$, and α^j_i is the ith component of Shapley value of subgame G^j_{co} using characteristic function $V^j(\cdot)$, $j = 1, 2$, $i \in N$. Then taking into account vector of initial distribution π^0 determine the allocation $\overline{\alpha}$ in G_{co} by Definition 8:

$$\overline{\alpha} = (\overline{\alpha}_1, \overline{\alpha}_2) = (658.20, 491.80).$$

Verify the principles of stable cooperation. To satisfy the principle of subgame consistency we should calculate the PDP for the allocation $\overline{\alpha}$ equals to $\pi^0\alpha$ using (16):

$$\beta_1 = \begin{pmatrix} 7.08 \\ 6.08 \end{pmatrix}, \quad \beta_2 = \begin{pmatrix} 5.92 \\ 3.92 \end{pmatrix}.$$

As we can see not all the components of PDP are equal to the corresponding payoffs of players in the states. So if players get their payoffs determined by the initial rules of the game they can't receive the components of the chosen allocation (in the mathematical expectation sense). It tells about subgame inconsistency of chosen cooperative agreement. Realize α-regularization of initial stochastic game G using PDP and Definition 3. In α-regularization of game G players' payoffs in state Γ^1 are as follows:

$$\begin{pmatrix} (2;2) & (7.08;5.92) \\ (11;3) & (5;4) \end{pmatrix},$$

and in state Γ^2 players' payoffs look like:

$$\begin{pmatrix} (3;1) & (2;7) \\ (5.92;3.92) & (5;3) \end{pmatrix}.$$

Check the second principle of stable cooperation which is strategic stability of cooperative agreement. For this purpose verify the truth of inequality (20) from Theorem 1. Compute $F(S)$ for all $S \subset N$ and obtain that $F(\{1\}) = (500.00, 498.00)$, $F(\{2\}) = (332.44, 332.78)$. Then inequality (20) is equivalent to the following ones:

$$\beta_1 = \begin{pmatrix} 7.08 \\ 6.08 \end{pmatrix} \geqslant \begin{pmatrix} 5.59 \\ 4.39 \end{pmatrix} = (\mathbb{I} - \delta \Pi(\overline{\eta}))F(\{1\}),$$

$$\beta_2 = \begin{pmatrix} 5.92 \\ 3.92 \end{pmatrix} \geqslant \begin{pmatrix} 3.22 \\ 3.43 \end{pmatrix} = (\mathbb{I} - \delta \Pi(\overline{\eta}))F(\{2\}).$$

They are true and we can say about strategic stability of the players' chosen cooperative agreement.

Verify the third principle condition of irrational behavior proofness. The sufficient condition for this principle from Proposition 1 holds as you can see here:

$$(\mathbb{I} - \delta \Pi(\overline{\eta}))(\alpha_1 - V(\{1\})) = \begin{pmatrix} 2.08 \\ 1.08 \end{pmatrix} \geqslant 0,$$

$$(\mathbb{I} - \delta \Pi(\overline{\eta}))(\alpha_2 - V(\{2\})) = \begin{pmatrix} 2.08 \\ 1.08 \end{pmatrix} \geqslant 0.$$

For this numerical example we have made the α-regularization of the initial game G, and checked the principles of stable cooperation, they all are satisfied.

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Decision Making under Many Quality Criteria

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Abstract We mean a quality criterion as a function from a set of alternatives in some chain (i.e. linearly ordered set). Decision making under many quality criteria is considered. We assume that some rule for preferences is fixed and it leads to a partial ordering on the set of alternatives. We study a problem of construction of generalized criterion for models of decision making under many quality criteria. The main result is connected with finding of additional information under which a general criterion is unique up to a natural equivalence.

Keywords: Multi-criteria decision making, quality criteria, general criterion, direction map.

1. Introduction

A general model of multi-criteria decision making can be presented in the form of a system

$$\langle A, q_1, \ldots, q_m \rangle, \tag{1}$$

where A is a set of all alternatives (or outcomes) and $q_1, \ldots, q_m$ are criteria for valuation of these alternatives. We consider decision making under certainty; then alternatives and outcomes coincide. Formally each criterion $q_j,\ j \in J = \{1, \ldots, m\}$ is a function from the set A in some scale points of which are results for measurement of criterion q_j. Recall that every scale has some set of *acceptable transformations* and the measurement produced up a some acceptable transformation.

Definition 1. A criterion q_j is called *a quality one* if its scale is some linearly ordered set $\langle C_j, \sigma_j \rangle$, i.e. a chain (concepts and notations connected with ordered sets, see in Birkhoff (1967)). In this case acceptable transformations are all isotonic functions defined on C_j.

We assume that for a class of models of the kind (12) *some rule for preferences* is fixed. Any rule for preferences leads to construction of certain preference relation ω on the set of alternatives A. The most important rule for preferences are *Pareto-dominance* $\leq^{Pd}$ and *modified Pareto-dominance* $<^{mPd}$ which are defined, respectively, by formulas:

$$a_1 \leq^{Pd} a_2 \Leftrightarrow (\forall j \in J)\, q_j\,(a_1) \overset{\sigma_j}{\leqq} q_j\,(a_2)\,, \tag{2}$$

$$a_1 <^{mPd} a_2 \Leftrightarrow (\forall j \in J)\, q_j\,(a_1) \overset{\sigma_j}{<} q_j\,(a_2)\,. \tag{3}$$

Definition 2. A pair $\langle A, \omega \rangle$, where A is a set of alternatives and ω is a preference relation on A is called a *space of preferences.*

Since the preference relation ω defined by (2) or (3) is a partial order relation, further we will consider space of preferences $\langle A, \omega \rangle$ as a (partial) ordered set, that is ω satisfies to axiom reflexivity, transitivity and anti-symmetry. An equivalence relation ε is said to be *stable* in a partial ordered set $\langle A, \omega \rangle$ if there exists isotonic function f from $\langle A, \omega \rangle$ into some chain $\langle B, \sigma \rangle$ such that ε is the kernel of f. In the section 2 we investigate a structure of stable equivalences in arbitrary ordered set. Some specification of stable equivalence leads to notion of map in ordered set.

The section 3 contains basic results concerning of model of decision making with many quality criteria. Here we study a problem of construction of general criterion for such models. We introduce a notion of *direction map* as a map which defines a general criterion up to the natural equivalence. The main results of this article are Theorem 3 and Theorem 4 in which necessary and sufficient conditions for map and for direction map are given. The existence of direction map for arbitrary ordered set is state also. In section 4 we consider some examples for construction of direction maps in ordered set and corresponding embeddings of ordered set into a chain.

2. Stable equivalences in ordered sets

2.1. Kernels of isotonic functions

Consider an arbitrary partially ordered set $\langle A, \omega \rangle$ and let ε be equivalence on A. Recall that factor-relation ω/ε is a binary relation on factor-set A/ε defined by

$$C_1 \overset{\omega/\varepsilon}{\leqq} C_2 \Leftrightarrow a_1 \overset{\omega}{\leqq} a_2 \text{ for some } a_1 \in C_1, a_2 \in C_2 \, (C_1, C_2 \in A/\varepsilon) \,. \tag{4}$$

Definition 3. An equivalence ε is called *stable* in ordered set $\langle A, \omega \rangle$ if the factor-relation ω/ε is acyclic.

The kernel of arbitrary function $f\colon A \to B$ is an equivalence relation ε_f on A defined as follows: $\varepsilon_f = \left\{(a_1, a_2) \in A^2 \colon f(a_1) = f(a_2)\right\}$.

Let $\langle A, \omega \rangle$ and $\langle B, \sigma \rangle$ be two ordered sets. A function $f\colon A \to B$ is called *isotonic* one if the condition

$$a_1 \overset{\omega}{\leqq} a_2 \Rightarrow f(a_1) \overset{\sigma}{\leqq} f(a_2) \tag{5}$$

holds.

For given ordered set, a characterization of kernels of its isotonic functions is given by the following theorem.

Theorem 1. *Let $\langle A, \omega \rangle$ be an arbitrary ordered sets and $\varepsilon \subseteq A^2$ be equivalence relation on A. Equivalence ε coincides with kernel of isotonic function from $\langle A, \omega \rangle$ in some ordered set $\langle B, \sigma \rangle$ if and only if ε is stable in $\langle A, \omega \rangle$.*

Proof (of theorem 1). Necessity. At first remark that acyclic condition for factor-relation ω/ε means the following implication

$$a_0 \overset{\omega}{\leqq} a_1' \overset{\varepsilon}{\equiv} a_1 \overset{\omega}{\leqq} \ldots \overset{\varepsilon}{\equiv} a_n \overset{\omega}{\leqq} a_0' \overset{\varepsilon}{\equiv} a_0 \Rightarrow a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_n \tag{6}$$

for any natural n.

Suppose equivalence $\varepsilon \subseteq A^2$ coincides with a kernel of an isotonic function f from $\langle A, \omega \rangle$ in some ordered set $\langle B, \sigma \rangle$ i.e. $\varepsilon = \varepsilon_f$. If the assumption of implication (6) holds then using isotonic condition (5) we have

$$f(a_0) \overset{\sigma}{\leqq} f(a_1') = f(a_1) \overset{\sigma}{\leqq} \ldots = f(a_n) \overset{\sigma}{\leqq} f(a_0') = f(a_0)$$

hence by acyclic of order σ the equality $f(a_0) = f(a_1) = \ldots = f(a_n)$ holds, that is $a_0 \overset{\varepsilon}{\equiv} a_1 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_n$.

Sufficiency. Let an equivalence ε be stable, i.e. the factor-relation ω/ε is acyclic one. In this case, its transitive closure $Tr(\omega/\varepsilon)$ is an order relation on factor-set A/ε and the canonical function $f_\varepsilon \colon A \to A/\varepsilon$ is an isotonic one from $\langle A, \omega \rangle$ into $\langle A/\varepsilon, Tr(\omega/\varepsilon) \rangle$. Since the kernel of f_ε is ε, the sufficient condition is proved. □

2.2. A structure of stable equivalences in ordered set

It follows from (6) that the intersection of any family of stable equivalences in ordered set $\langle A, \omega \rangle$ is a stable equivalence also; since the universal relation A^2 is stable, we obtain a closure operation E_s on the set of all subsets of A^2. For any binary relation $\rho \subseteq A^2$, $E_s(\rho)$ is the intersection of all stable equivalences ε with $\varepsilon \supseteq \rho$. In other words $E_s(\rho)$ is the smallest stable equivalence which contains ρ, it named a *stable equivalent closure* of ρ. The set of all stable equivalences in ordered set $\langle A, \omega \rangle$ forms a complete lattice. We wish to find an evident form for operations of infimum and supremum in this lattice. As the first step, we show a construction for $E_s(\rho)$ where ρ is an arbitrary binary relation on A. Since $E_s(\rho)$ coincides with $E_s(\varepsilon)$, where ε is the equivalence closure of ρ, it is sufficiently to find $E_s(\varepsilon)$ for arbitrary equivalence $\varepsilon \subseteq A^2$. In the case the factor-relation ω/ε is acyclic, equivalence ε is stable and $E_s(\varepsilon) = \varepsilon$. In the opposite case the factor-relation ω/ε contains some cycles (contours). It is well known that the identification of cycles leads to relation (or graph) without of cycles (see Zykov (1969)). Formally "the identification of cycles" of arbitrary relation ρ is its factorization by equivalence $AR(\rho) = Tr(\rho) \bigcap Tr(\rho^{-1})$. Thus we need here in a double factorization: the first step is the factorization of order relation ω under equivalence ε and the second step is the factorization of the factor-relation ω/ε under the equivalence $AR(\omega/\varepsilon)$. Since double factorization can be reduced to one factorization, we have the following assertion.

Lemma 1. *Let $\langle A, \omega \rangle$ be an ordered set and ε be an equivalence on A. Then stable equivalent closure $E_s(\varepsilon)$ of ε is an equivalence, classes of which are unions of ε-classes belonging to one cycle of factor-relation ω/ε.*

Using Lemma 1, it is easy to show that the condition $a_1 \overset{E_s(\varepsilon)}{\equiv} a_2$ means the existence of cycle in graph $\langle A, \omega \cup \varepsilon \rangle$ which contains elements a_1 and a_2. Since classes of equivalence $E_s(\varepsilon)$ coincide with cycles of the relation $\omega \cup \varepsilon$, we have

$$E_s(\varepsilon) = AR(\omega \cup \varepsilon). \tag{7}$$

Corollary 1. *An equivalence ε is stable in ordered set $\langle A, \omega \rangle$ if and only if $AR(\omega \cup \varepsilon) = \varepsilon$.*

Remark 1. It is well known that there exists a simple algorithm for finding of cycles of graph; there is an algorithm of construction of equivalence $AR(\rho)$ for arbitrary relation ρ (see, for example, Zykov (1969)). By using this algorithm, we have according formula (7) a method for constructing of stable equivalent closure of an equivalence $\varepsilon \subseteq A^2$.

Consider now a problem of construction of stable equivalent closure for arbitrary relation $\rho \subseteq A^2$. Let $E(\rho)$ be equivalent closure of a relation ρ. It is clear that stable equivalent closure of ρ and $E(\rho)$ coincide: $E_s(\rho) = E_s(E(\rho))$. On the other hand it is easy to show that the existence of a cycle in a graph $\langle A, \omega \cup E(\rho)\rangle$, which contains elements $a', a'' \in A$, means the existence of cycle in graph $\left\langle A, \omega \cup \rho \cup \rho^{-1}\right\rangle$ containing these elements, hence $AR(\omega \cup E(\rho)) = AR\left(\omega \cup \rho \cup \rho^{-1}\right)$. By using (7), we obtain the following equality for equivalent stable closure in ordered set $\langle A, \omega\rangle$:

$$E_s(\rho) = AR\left(\omega \cup \rho \cup \rho^{-1}\right).$$

Summarizing results of this section we have the following assertion.

Theorem 2. *Let $\langle A, \omega\rangle$ be an arbitrary ordered set. Then the set of all stable equivalences in $\langle A, \omega\rangle$ forms a complete lattice in which operations of infimum and supremum can be represented as follows:*

$$\inf_{i \in I}(\varepsilon_i) = \bigcap_{i \in I} \varepsilon_i,$$

$$\sup_{i \in I}(\varepsilon_i) = E_s\left(\bigcup_{i \in I} \varepsilon_i\right) = AR\left(\omega \cup \bigcup_{i \in I} \varepsilon_i\right).$$

3. Generalized criterion for decision making with many quality criteria

3.1. Maps and direction maps in ordered set

The problem of construction of a generalized criterion is the main problem of multi-criteria optimization. For models of decision making with many quality criteria, a generalized criterion can be defined as embedding of the space of preferences $\langle A, \omega\rangle$ associated with given decision making problem into some chain $\langle C, \sigma\rangle$ which is a scale for the generalized criterion. Our basic idea for constructing of generalized criterion is that we introduce some additional information under which a generalized criterion became unique up to natural equivalence.

Firstly we consider some preliminary notions.

Definition 4. An embedding of ordered set $\langle A, \omega\rangle$ into chain $\langle C, \sigma\rangle$ is called a strict isotonic function, i.e. a function $\varphi\colon A \to C$ with condition

$$a_1 \overset{\omega}{<} a_2 \Rightarrow \varphi(a_1) \overset{\sigma}{<} \varphi(a_2). \tag{8}$$

Remark 2. Let $\langle A, \omega\rangle$ be an arbitrary ordered set and $\varphi\colon A \to C$ its embedding in some chain $\langle C, \sigma\rangle$. Then we can define a linear quasi-ordering ω_φ by the formula:

$$a_1 \overset{\omega_\varphi}{\leqq} a_2 \Leftrightarrow \varphi(a_1) \overset{\sigma}{\leqq} \varphi(a_2). \tag{9}$$

Thus we receive a linear quasi-order on A which preserves the strict order $\overset{\omega}{<}$ (that is the condition $a_1 \overset{\omega}{<} a_2$ implies $a_1 \overset{\omega_\varphi}{<} a_2$). The relation ω_φ is said to be a linear quasi-ordering induced by embedding φ.

Definition 5. Let $\langle A, \omega \rangle$ be an ordered set, $\varepsilon \subseteq A^2$ — equivalence on A. A partition of A with classes of ε–equivalent elements is called a *map in* $\langle A, \omega \rangle$ if there exists embedding φ from $\langle A, \omega \rangle$ into some chain $\langle C, \sigma \rangle$ whose kernel is ε (i.e. $\varepsilon_\varphi = \varepsilon$).

Definition 6. Two embeddings φ_1, φ_2 of ordered set $\langle A, \omega \rangle$ are said to be naturally equivalent (in notation: $\varphi_1 \overset{nat}{\sim} \varphi_2$) if for any $a_1, a_2 \in A$ the following condition

$$\varphi_1(a_1) \overset{\sigma}{\leqq} \varphi_1(a_2) \Leftrightarrow \varphi_2(a_1) \overset{\sigma}{\leqq} \varphi_2(a_2) \tag{10}$$

holds. According to Remark 2, the condition: $\varphi_1 \overset{nat}{\sim} \varphi_2$ means that linear quasi-orderings of the set A induced by embeddings φ_1 and φ_2 coincide, that is, $\omega_{\varphi_1} = \omega_{\varphi_2}$.

Definition 7. Let $\langle A, \omega \rangle$ be an ordered set, $\varepsilon \subseteq A^2$ — equivalence on A. A partition of A with classes of ε–equivalent elements is called a *direction map in* $\langle A, \omega \rangle$ if

1) there exists an embedding φ of $\langle A, \omega \rangle$ into some chain whose kernel is ε (i.e. this partition is a map) and

2) any two embeddings of ordered set $\langle A, \omega \rangle$ with kernel ε are naturally equivalent.

Definition 8. An embedding φ of ordered set $\langle A, \omega \rangle$ into some chain whose kernel is a direction map is called a *direction embedding*. For direction embedding φ the following important property holds: if g is any embedding of ordered set $\langle A, \omega \rangle$ with $\varepsilon_g = \varepsilon_\varphi$ then $\omega_g = \omega_\varphi$. Thus quasi-orderings of the set of alternatives induced by a direction embeddings with fix kernel are the same. At this reason we consider direction embeddings as generalized criteria for decision making with many quality criteria.

3.2. Characterization theorems for maps and direction maps

We now consider two main problems connected with construction of generalized criteria for models of decision making with many quality criteria: a characterization of maps and direction maps.

Lemma 2. *Let $\langle A, \omega \rangle$ and $\langle B, \sigma \rangle$ be ordered sets and $\varphi \colon A \to B$ an isotonic function. The function φ is strict isotonic if and only if each class of the kernel ε_φ is a discrete subset (that is, antichain).*

Proof (of lemma 3.2). Necessity. Let φ be a strict isotonic function from $\langle A, \omega \rangle$ into $\langle B, \sigma \rangle$. Suppose $a_1 \overset{\varepsilon_\varphi}{\equiv} a_2$ and $a_1 \overset{\omega}{\leqq} a_2$. We need to show $a_1 = a_2$. In the opposite case $a_1 \overset{\omega}{<} a_2$ holds and since φ is strict isotonic, we have $\varphi(a_1) \overset{\sigma}{<} \varphi(a_2)$; on the other hand according with $a_1 \overset{\varepsilon_\varphi}{\equiv} a_2$ we obtain $\varphi(a_1) = \varphi(a_2)$ in contradiction with preceding condition.

Sufficiency. Suppose $a_1 \overset{\omega}{<} a_2$. Since φ is isotonic, we obtain $\varphi(a_1) \overset{\sigma}{\leqq} \varphi(a_2)$. The assumption $\varphi(a_1) = \varphi(a_2)$ implies $a_1 \overset{\varepsilon_\varphi}{\equiv} a_2$ and together with $a_1 \overset{\omega}{\leqq} a_2$, we have according to discrete condition the equality $a_1 = a_2$ in contradiction with $a_1 \overset{\omega}{<} a_2$. □

Theorem 3 (a characterization of maps). *Let $\langle A, \omega \rangle$ be an ordered set and ε be an equivalence on A. The partition with classes of ε–equivalent elements is a map in $\langle A, \omega \rangle$ if and only if equivalence ε is stable and each its class is a discrete subset.*

Proof (of theorem 3). Necessity. Suppose the partition with classes of ε–equivalent elements is a map in $\langle A, \omega\rangle$, that is, there exists a chain $\langle C, \sigma\rangle$ and an embedding φ of $\langle A, \omega\rangle$ into $\langle C, \sigma\rangle$ with $\varepsilon_\varphi = \varepsilon$. Since $\varepsilon = \varepsilon_\varphi$ is a kernel of isotonic function φ, according to Theorem 1, ε is stable. It follows from Lemma 3.2 that each class of ε–equivalent elements is a discrete subset.

Sufficiency. Since an equivalence ε is stable, the factor-relation ω/ε is acyclic hence its transitive closure $Tr\,(\omega/\varepsilon)$ is an order relation on the factor set A/ε and the canonical function $f_\varepsilon\colon A \to A/\varepsilon$ is an isotonic function from $\langle A, \omega\rangle$ into $\langle A/\varepsilon, Tr\,(\omega/\varepsilon)\rangle$. According to Lemma 3.2 this function is strict isotonic one. Let σ be a linear order on the factor-set A/ε with $\sigma \supseteq Tr\,(\omega/\varepsilon)$ (the existence of such linear order follows from known Szpilrajns Theorem). Then f_ε is a strict isotonic function from $\langle A, \omega\rangle$ into linear ordered set $\langle A/\varepsilon, \sigma\rangle$, that is, an embedding of ordered set $\langle A, \omega\rangle$ into some chain and its kernel is ε. $\square$

The main result of this article states the following

Theorem 4 (a characterization of direction maps). *Let $\langle A, \omega\rangle$ be an ordered set and ε be an equivalence on A. The partition with classes of ε–equivalent elements is a direction map in $\langle A, \omega\rangle$ if and only if ε is a maximal between stable equivalences whose classes are discrete subsets.*

A proof of this theorem is based on the following two lemmas.

Lemma 3. *Given an ordered set $\langle A, \omega\rangle$. An equivalence ε on A is a maximal between stable equivalences whose classes are discrete subsets if and only if the transitive closure of factor-relation ω/ε is a linear order on factor-set A/ε (that is, the ordered set $\langle A/\varepsilon, Tr\,(\omega/\varepsilon)\rangle$ is a chain).*

Proof (of lemma 3). Necessity. Let an equivalence $\varepsilon \subseteq A^2$ is a maximal between stable equivalences whose classes are discrete subsets. Since $Tr\,(\omega/\varepsilon)$ is an order relation for any stable equivalence ε, we need to proof the linearity condition only. Fix two classes $C', C'' \in A/\varepsilon$. It can be the following two cases: 1). There exist elements $a' \in C', a'' \in C''$ which are comparable under the order ω. 2). Any two elements of this classes are uncomparable under the order ω. It is evident that in the first case, the classes C' and C'' are comparable under the order $Tr\,(\omega/\varepsilon)$. We now check a comparability of these classes in the second case. Consider the equivalence $\bar\varepsilon$ one of the classes whose is $C' \cup C''$ and other classes are the same as for equivalence ε. Obviously $\bar\varepsilon \supset \varepsilon$ and using the assumption 2) we have that all classes of $\bar\varepsilon$ are discrete subsets in $\langle A, \omega\rangle$. Then according to maximality condition the equivalence ε is not stable i.e. there is a cycle in the graph $\langle A/\bar\varepsilon, \omega/\bar\varepsilon\rangle$:

$$\left(\bar C_0, \bar C_1\right) \in \omega/\bar\varepsilon, \left(\bar C_1, \bar C_2\right) \in \omega/\bar\varepsilon, \ldots, \left(\bar C_{s-1}, \bar C_s\right) \in \omega/\bar\varepsilon, \left(\bar C_s, \bar C_0\right) \in \omega/\bar\varepsilon \qquad (11)$$

and at least one pair of neighbour elements are different. Not less of generality we assume that all classes in (11) except the first and the last members are different. Evidently (11) contains the class $C' \cup C''$ (in the opposite case, we have a contradiction with stable condition of equivalence ε). By setting $\bar C_k = C' \cup C''$ and using that for any $i = 0, \ldots, s$, $i \neq k$, $\bar C_i$ is a class of equivalence ε, we obtain from (11) the condition

$$a_0 \overset{\omega}{\leqq} a_1' \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_{k-1} \overset{\omega}{\leqq} a_k' \overset{\bar\varepsilon}{\equiv} a_k \overset{\omega}{\leqq} a_{k+1}' \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_s \overset{\omega}{\leqq} a_0' \overset{\varepsilon}{\equiv} a_0, \qquad (12)$$

where $a_0, a'_0 \in \bar{C}_0, \ldots, a_s, a'_s \in \bar{C}_s$. Consider elements $a_k, a'_k \in \bar{C}_k = C' \cup C''$. The assumption these elements belong to one of the class (C' or C'') implies the existence of cycle in graph $\langle A/\varepsilon, \omega/\varepsilon\rangle$ that impossible. Suppose $a_k \in C'$, $a'_k \in C''$. Then from (12) we have $\left([a_0]_\varepsilon, [a'_k]_\varepsilon\right) \in Tr\,(\omega/\varepsilon)$, $([a_k]_\varepsilon, [a_0]_\varepsilon) \in Tr\,(\omega/\varepsilon)$, hence $\left([a_k]_\varepsilon, [a'_k]_\varepsilon\right) \in Tr\,(\omega/\varepsilon)$ i.e. $C' \overset{Tr(\omega/\varepsilon)}{\leqq} C''$ which was to be proved.

Sufficiency. Assume the transitive closure of factor-relation ω/ε is a linear order on factor-set A/ε. Consider a stable equivalence $\bar{\varepsilon}$ in ordered set $\langle A, \omega\rangle$ with $\bar{\varepsilon} \supset \varepsilon$. Then there exists such a pair of elements $a, b \in A$ that $a \overset{\bar{\varepsilon}}{\equiv} b$ is truth and $a \overset{\varepsilon}{\equiv} b$ is false. Because the order $Tr\,(\omega/\varepsilon)$ is linear, $f_\varepsilon(a) \overset{Tr(\omega/\varepsilon)}{\leqq} f_\varepsilon(b)$ or $f_\varepsilon(b) \overset{Tr(\omega/\varepsilon)}{\leqq} f_\varepsilon(a)$ holds. Suppose the first correlation is truth. Put $f_\varepsilon(a) = C$, $f_\varepsilon(b) = C'$. According to definition of transitive closure there exists a finite sequence of ε–classes: $C = C_0, C_1, \ldots, C_m = C'$ such that $(C_i, C_{i+1}) \in \omega/\varepsilon$ for all $i = 0, \ldots, m-1$. According with definition of factor-relation it means that

$$a_0 \overset{\omega}{\leqq} a'_1 \overset{\varepsilon}{\equiv} a_1 \overset{\omega}{\leqq} a'_2 \overset{\varepsilon}{\equiv} \ldots \overset{\varepsilon}{\equiv} a_{m-1} \overset{\omega}{\leqq} a'_m \tag{13}$$

holds for some elements $a_0 \in C_0; a_1, a'_1 \in C_1; \ldots; a_{m-1}, a'_{m-1} \in C_{m-1}$, $a'_m \in C_m$.

In (13) the strict inequality $a_k \overset{\omega}{<} a'_{k+1}$ holds at least for one $k = 0, \ldots, m-1$ (in the opposite case we have $a_0 \overset{\varepsilon}{\equiv} a'_m$ and using correlations $a \overset{\varepsilon}{\equiv} a_0$, $b \overset{\varepsilon}{\equiv} a'_m$ we obtain $a \overset{\varepsilon}{\equiv} b$ in contradiction with our assumption). On the other hand since $\varepsilon \subset \bar{\varepsilon}$ and $a_0 \overset{\bar{\varepsilon}}{\equiv} a \overset{\bar{\varepsilon}}{\equiv} b \overset{\bar{\varepsilon}}{\equiv} a'_m$ then $a_0 \overset{\bar{\varepsilon}}{\equiv} a'_m$ and from (13) it follows

$$a_0 \overset{\omega}{\leqq} a'_1 \overset{\bar{\varepsilon}}{\equiv} a_1 \overset{\omega}{\leqq} a'_2 \overset{\bar{\varepsilon}}{\equiv} \ldots \overset{\bar{\varepsilon}}{\equiv} a_{m-1} \overset{\omega}{\leqq} a'_m \overset{\bar{\varepsilon}}{\equiv} a_0 \tag{14}$$

We obtain from (14) by using the stable condition for equivalence $\bar{\varepsilon}$: $a \overset{\bar{\varepsilon}}{\equiv} a_0 \overset{\bar{\varepsilon}}{\equiv} a_1 \overset{\bar{\varepsilon}}{\equiv} a'_1 \overset{\bar{\varepsilon}}{\equiv} \ldots \overset{\bar{\varepsilon}}{\equiv} a_{m-1} \overset{\bar{\varepsilon}}{\equiv} a'_{m-1} \overset{\bar{\varepsilon}}{\equiv} a'_m$ hence $a_k \overset{\bar{\varepsilon}}{\equiv} a'_{k+1}$. Because the strict inequality $a_k \overset{\omega}{<} a'_{k+1}$ holds (see above), we have that the class $[a]_{\bar{\varepsilon}}$ is not discrete one. □

Lemma 4. *Let A be an arbitrary set, ρ_1, ρ_2 be linear quasi-orderings on A and $\varepsilon_{\rho_1} = \rho_1 \cap \rho_1^{-1}$, $\varepsilon_{\rho_2} = \rho_2 \cap \rho_2^{-1}$ their kernels. Then conditions $\rho_1 \subseteq \rho_2$ and $\varepsilon_{\rho_1} = \varepsilon_{\rho_2}$ imply $\rho_1 = \rho_2$.*

Proof (of lemma 4). Suppose the strict inclusion $\rho_1 \subset \rho_2$ holds. Then there exists a pair of elements $(a_1, a_2) \in \rho_2 \setminus \rho_1$. Because $(a_1, a_2) \notin \rho_1$ we have $(a_2, a_1) \in \rho_1$ according to linearity condition and $(a_2, a_1) \in \rho_2$. We obtain $(a_1, a_2) \in \rho_2 \cap \rho_2^{-1} = \varepsilon_{\rho_2} = \varepsilon_{\rho_1} \subseteq \rho_1$ hence $(a_1, a_2) \in \rho_1$ in contradiction with our assumption. □

Proof (of theorem 4). Necessity. Suppose the partition with classes of ε–equivalent elements is a direction map in $\langle A, \omega\rangle$. Then according to Theorem 3 equivalence ε is stable and each its class is a discrete subset. It is remains to be proved the maximality condition. Assume that the maximality condition does not hold for equivalence ε. Then according to Lemma 3, the order relation $Tr\,(\omega/\varepsilon)$ is not a linear one on factor-set A/ε hence there exist two classes $C', C'' \in A/\varepsilon$ which are not comparable under $Tr\,(\omega/\varepsilon)$. Let σ_1 and σ_2 be two linear orderings of A/ε containing the order $Tr\,(\omega/\varepsilon)$ such that $C' \overset{\sigma_1}{<} C''$ and $C'' \overset{\sigma_2}{<} C'$ (the existence of such orderings follows from well known Szpilrajns Theorem). Consider two chain $\langle A/\varepsilon, \sigma_1\rangle$ and $\langle A/\varepsilon, \sigma_2\rangle$. Let $f_k\colon A \to A/\varepsilon$ be the canonical function from the ordered set $\langle A, \omega\rangle$ into $\langle A/\varepsilon, \sigma_k\rangle$, $k = 1, 2$. It is shown above that f_k is an embedding of the ordered set $\langle A, \omega\rangle$ into the chain $\langle A/\varepsilon, \sigma_k\rangle$, $k = 1, 2$, and the kernel of function f_k is ε (see the proof of sufficiency in Theorem 3). Fix arbitrary elements $a_1 \in C'$ and $a_2 \in C''$. According to definition of linear quasi-ordering induced by an embedding (see Remark 2) we have: $a_1 \overset{\omega_{f_1}}{<} a_2$ but $a_2 \overset{\omega_{f_2}}{<} a_1$. Thus linear quasi-orderings ω_{f_1} and ω_{f_2} are different hence embeddings f_1 and f_2 are not naturally equivalent which was to be proved.

Sufficiency. Let ε be equivalence satisfying conditions of Theorem 4. According to Theorem 3, ε is a map in ordered set $\langle A, \omega\rangle$. It is remains to be proved that ε is a direction map. Consider an embedding $g\colon A \to C$ of ordered set $\langle A, \omega\rangle$ into some chain $\langle C, \sigma\rangle$ with $\varepsilon_g = \varepsilon$. Denote by ρ_g the linear quasi-ordering on A induced by embedding g and by ρ_0 the linear quasi-ordering on A induced by embedding f_ε. We need to proof $\rho_g = \rho_0$. Since kernels of these functions are coincide, according to Lemma 4 it is sufficiently to check the inclusion $\rho_0 \subseteq \rho_g$. Indeed assume $(a', a'') \in \rho_0$ i.e. $(f_\varepsilon(a'), f_\varepsilon(a'')) \in Tr(\omega/\varepsilon)$. By definition of transitive closure there exists a finite consequence of elements $a_0, a_1', a_1, \ldots, a_{m-1}', a_{m-1}, a_m'$ such that

$$a' \overset{\varepsilon}{\equiv} a_0 \overset{\omega}{\leqq} a_1' \overset{\varepsilon}{\equiv} a_1 \overset{\omega}{\leqq} \ldots \overset{\varepsilon}{\equiv} a_{m-1} \overset{\omega}{\leqq} a_m' \overset{\varepsilon}{\equiv} a''. \tag{15}$$

Because the function g is isotonic and its kernel is ε, we obtain from (15):

$$g(a') = g(a_0) \overset{\sigma}{\leqq} g(a_1') = g(a_1) \overset{\sigma}{\leqq} \ldots = g(a_{m-1}) \overset{\sigma}{\leqq} g(a_m') = g(a'')$$

hence $g(a') \overset{\sigma}{\leqq} g(a'')$ that is $(a', a'') \in \rho_g$. The inclusion $\rho_0 \subseteq \rho_g$ is shown and according to Lemma 4 we have $\rho_g = \rho_0$ which completes the proof of Theorem 4.

□

Corollary 2 (the existence of direction map). *For any ordered set there exists a direction map.*

Proof (of corollary 2). Let $\langle A, \omega\rangle$ be an arbitrary ordered set. According to Theorem 4 it sufficiently to show that there exists a maximal stable equivalence in $\langle A, \omega\rangle$, whose classes are discrete subsets. Using Zorns Lemma, we need to check the following condition:

(i) *Any chain of stable equivalences with discrete classes in $\langle A, \omega\rangle$ has a majorant.*

Indeed, let $(\varepsilon_i)_{i\in I}$ be a chain of stable equivalences with discrete classes in $\langle A, \omega\rangle$. It is easy to show that a binary relation $\varepsilon = \cup_{i\in I}\varepsilon_i$ is an equivalence with discrete classes also. It is follows from (6) that equivalence ε is a stable one. Further, stable equivalences with discrete classes in $\langle A, \omega\rangle$ exist always, for example the identity equivalence Δ_A. According with Zorns Lemma, we have the inclusion $\Delta_A \subseteq \bar{\varepsilon}$, where $\bar{\varepsilon}$ is an maximal stable equivalence with discrete classes in $\langle A, \omega\rangle$. Thus a required equivalence is found. Some methods for construction of direction maps for finite ordered sets will be considered in the next section. □

4. Examples

Example 1. Let $\langle A, \omega \rangle$ be a finite ordered set. Recall that *the height* $d(x)$ of an element $x \in A$ means the maximum length d of chains in $\langle A, \omega \rangle$ of the form $x_0 < x_1 < \ldots < x_d = x$ having x for greatest element (see Birkhoff (1967), p. 11).

Lemma 5. *The function of height d is a direction embedding of finite ordered set into the chain $\mathbb{N}$ of natural numbers.*

Proof (of lemma 5). It is follows from the definition that the function of height d is a strict isotonic one, that is, an embedding of ordered set $\langle A, \omega \rangle$ into chain N. It is remain to be shown the embedding is direction one. Using Theorem 4, it is sufficiently to check that union of any two classes of equivalence ε_d – the kernel of the function d – is not a discrete subset in $\langle A, \omega \rangle$. Indeed, let C_1 and C_2 be two classes of ε_d; put $d(a) = n_1, d(b) = n_2$ for all $a \in C_1, b \in C_2$ and $n_1 < n_2$. By definition of the height for element $b \in C_2$ there exists a sequence of the form $x_0 < \ldots < x_{n_1} < \ldots < x_{n_2} = b$ hence we have: $x_{n_1} \in C_1 \subseteq C_1 \cup C_2, x_{n_2} \in C_2 \subseteq C_1 \cup C_2$ thus $x_{n_1}, x_{n_2} \in C_1 \cup C_2$ and $x_{n_1} < x_{n_2}$ holds; subset $C_1 \cup C_2$ is not discrete. $\square$

Example 2. Let $\lambda(x)$ be a number of strict minorant for element x in finite ordered set $\langle A, \omega \rangle$. Obviously, the function λ is a strict isotonic one, that is, an embedding of ordered set $\langle A, \omega \rangle$ into chain $\mathbb{N}$. But in general case, this embedding is not direction one. Indeed, consider the ordered set presented its diagram in Fig. 1.

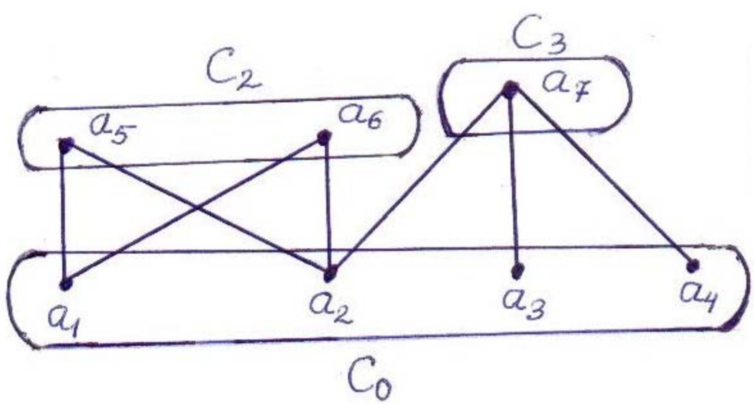

Figure1: Diagram of order

Here C_0, C_2, C_3 are classes of the equivalence ε, the kernel of function λ. Consider an equivalence ε_1 with classes $\{C_0, C_2 \cup C_3\}$. It is easy to see that ε_1 coincides with the kernel of strict isotonic function $g\colon A \to \{0, 1\}$, where $g(a_i) = 0$ for $a_i \in C_0$ and $g(a_j) = 1$ for $a_j \in C_2 \cup C_3$. According to results of section 3, ε_1 is stable equivalence with discrete classes. Because $\varepsilon_1 \supset \varepsilon$, the maximality condition does not hold for equivalence ε. By Theorem 4, the function λ is not a direction embedding.

Example 3. Consider a model of decision making with quality criteria which is given as follows. The set of alternatives is $A = \{a, b, c, d, e, f, g, h, k\}$; q_1, q_2, q_3 − criteria for evaluation of the alternatives; scales of these criteria are respectively:

$$Q_1 = \{\alpha_0 < \alpha_1 < \alpha_2\}, Q_2 = \{\beta_0 < \beta_1 < \beta_2\}, Q_3 = \{\gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 < \gamma_4\}.$$

Evaluations of alternatives under criteria q_1, q_2, q_3 are given by Table 1.

Table1: Evaluations of alternatives

A \ Q	q_1	q_2	q_3
a	α_0	β_0	γ_1
b	α_1	β_0	γ_0
c	α_2	β_0	γ_1
d	α_1	β_0	γ_2
e	α_2	β_1	γ_1
f	α_2	β_1	γ_2
g	α_1	β_2	γ_2
h	α_2	β_2	γ_2
k	α_1	β_2	γ_4

We will construct a direction map and corresponding linear quasi-ordering of alternatives for this model of decision making. For the first step, by using the Table 1, we define a preference relation ω in the form of Pareto-dominance (2). The order relation ω can be given by its diagram (see Fig. 2).

According to Lemma 5, the function of height d is a direction embedding of the ordered set $\langle A, \omega\rangle$ into the chain of natural numbers $\{0, 1, 2, 3, 4\}$. And the classes of equivalence ε, the kernel of the function d, define a direction map in ordered map $\langle A, \omega\rangle$. In our case, the classes of equivalence ε_d are: $C_0 = \{a, b\}, C_1 = \{c, d\}, C_2 = \{e, g\}, C_3 = \{f, k\}, C_4 = \{h\}$. A linear quasi-ordering of the set of alternatives A corresponding to the direction map is shown in Figure 3.

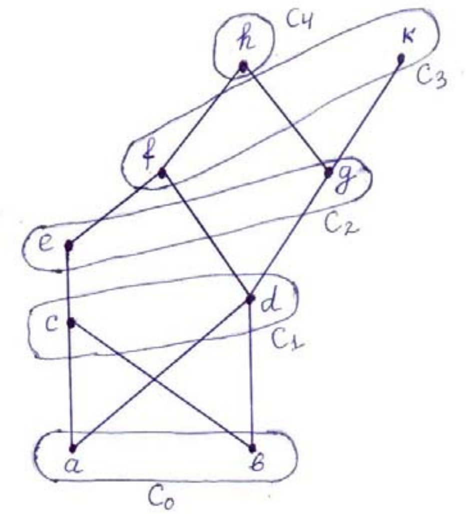

Figure2: Diagram of order ω

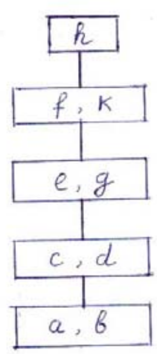

Figure3: A linear quasi-ordering of alternatives

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Solution for One-Stage Bidding Game with Incomplete Information*

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Abstract We investigate a model of one-stage bidding between two differently informed stockmarket agents for a risky asset (share). The random liquidation price of a share may take two values: the integer positive m with probability p and 0 with probability $1-p$. Player 1 (insider) is informed about the price, Player 2 is not. Both players know the probability p. Player 2 knows that Player 1 is an insider. Both players propose simultaneously their bids. The player who posts the larger bid buys one share from his opponent for this price. Any integer bids are admissible. The model is reduced to a zero-sum game with lack of information on one side. We construct the solution of this game for any p and m: we find the optimal strategies of both players and describe recurrent mechanism for calculating the game value. The results are illustrated by means of computer simulation.

Keywords: insider trading, asymmetric information, equalizing strategies, optimal strategies.

1. Introduction

The model of bidding for risky asset (a share) with different agent's information about liquidation value of a share was introduced by De Meyer and Saley, 2002.

A liquidation price of a share, which can take two values – high and low share price, depends on a random "state of nature". Before bidding starts a chance move determines the "state of nature" and therefore the liquidation value of a share. The probability p of choice of high share price is known to both players. Besides, Player 1 (insider) is informed about the "state of nature", Player 2 is not. Player 2 knows that Player 1 is an insider.

At each subsequent step both players simultaneously propose their bids for one share. The maximal bid wins and one share is transacted at this price. After this bids are reported to both players. Each player aims to maximize the value of his final portfolio (money plus liquidation value of obtained shares).

De Meyer and Saley reduce the model to a zero-sum repeated game with lack of information on the side of Player 2, they solve the game for any number of steps, find optimal behavior of both players and expected profit of insider.

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In De Meyer and Saley model any real bids are admissible.

It is more realistic to assume that players may assign only discrete bids proportional to a minimal currency unit. The models with admissible discrete bids are investigated by Domansky, 2007, **?**. These models are the models of n-stage bidding game $G_n^m(p)$ with incomplete information with high share price equal to integer positive m, low share price equal to zero, and admissible discrete bids. Solving of n-stage bidding game is combinatorially difficult. The solution was only obtained for the game $G_\infty^m(p)$ with unlimited duration.

The solution of finitely-stage games is an open problem. The solution has been found only for the difference between high and low share prices equal or less than 3 ($m \leq 3$) (Kreps, 2009b).

In this paper we give the complete solution for the one-stage bidding game $G_1^m(p)$ with arbitrary integer m and with any probability $p \in (0,1)$ of the high share price.

We develop recurrent approach to computing optimal strategies of uninformed player for any probability p based on analysis of structure of bids used in optimal strategies of both players. Non-strictly speaking, recursion is on the number of pure strategies used by Player 2 in optimal mixed strategy.

The optimal strategy of insider equalizes spectrum of obtained optimal strategy of Player 2, one can obtain distribution of it's weights solving the system of difference equations arizing from equalizing conditions. As a base for calculating solution elements we use solution on the ends of interval $[0,1]$: for probabilities p sufficiently close to 0 and 1 the game has solutions in pure strategies. When p runs from 0 (or 1) till some limit these pure strategies holds as optimal ones, and so on.

For the special finite set of p depending on m another approach to finding the solution of one-stage bidding game was suggested in the paper of Kreps, 2009a.

2. Model of one-stage bidding with arbitrary bids

Here we give the explicit solution for one-stage bidding game $G_1(p)$ with arbitrary admissible bids. We put $m = 1$ to make formulae more clear. It follows from the work of De Meyer and Saley, 2002 that the game value $V_1(p) = p(1-p)$.

Note that the optimal strategy of Player 1 is to post the bid zero at the state L and randomization of bids at the state H for any prior probability. This observation allows to reduce solving one-stage game $G_1(p)$ with incomplete information to solving the game on unit square with payoff function

$$K_p(x,y) = \begin{cases} (1-p)y + p(1-x), & \text{for } x > y; \\ (1-p)y, & \text{for } x = y; \\ (1-p)y - p(1-y), & \text{for } x < y. \end{cases}$$

Applying well-known heuristic methods of solving games on unit square with payoff function that has a break on the principal diagonal (Karlin, 1964) we find strategies of informed and uninformed players. Further in Theorem 1 it is shown that these strategies are optimal.

Denote by the same letter a mixed strategy of a player and corresponding cumulative density function: F_p is a mixed strategy of Player 1, G_p is a mixed strategy of Player 2. Write down the gain of Player 1 when he applies mixed strategy

F_p and Player 2 applies pure strategy y:

$$K_p(F_p, y) = (1-p)y - \int_0^y p(1-y)dF_p(x) + \int_y^1 p(1-x)dF_p(x) -$$

$$- \int_y^1 x dF_p(x) = (1-p)y - p(1-y)F_p(y) + p(1-F_p(y)) - \int_y^1 x dF_p(x).$$

The optimal strategy of Player 1 equalizes the payoffs for pure strategies in spectrums of optimal strategies of Player 2 (see Karlin, 1964).

We assume that optimal strategies of both players have the same spectrums. Putting the derivative by y of the function $K_p(F_p, y)$ equal to zero we obtain the differential equation for points of the common spectrum:

$$\frac{dF_p(x)}{dx} 2p(1-x) = pF_p(x) + (1-p). \tag{1}$$

It is easy to see that $F_p(0) = 0$. The equation (1) has the solution

$$F_p^*(x) = \frac{1-p}{p}((1-x)^{-1/2} - 1) \quad x \in [0, 1-(1-p)^2].$$

Applying similar reasoning for mixed strategy G_p of Player 2 we get:

$$K_p(x, G_p) = (1-p)\int_0^1 y dG_p(y) + p(1-x)G_p(x) \int_x^1 p(1-y)dG_p(y);$$

$$(1-y)\frac{dG_p(y)}{dy} = G_p(y). \tag{2}$$

Solving equation (2) and choosing the solution with the same spectrum as one's of solution of equation (1) we obtain

$$G_p^*(y) = \frac{1-p}{\sqrt{1-y}}, \text{ for } y \in [0, 1-(1-p)^2].$$

Observe that using this strategy Player 2 proposes a bid 0 with the positive probability $G_p^*(0) = 1-p$.

Theorem 1. *For the one-stage bidding game $G_1(p)$ the unique optimal strategy of Player 1 is to post* 0 *at the state L. At the state H this strategy is given by the cumulative depending on p density function on* $[0, 1]$

$$F_p^*(x) = \begin{cases} \frac{(1-p)(1-\sqrt{1-x})}{p\sqrt{1-x}}, \mathit{for}\ x \le (1-(1-p)^2); \\ 1, \qquad\qquad\qquad \mathit{for}\ x > (1-(1-p)^2). \end{cases}$$

The unique optimal strategy of Player 2 is given by the cumulative density function

$$G_p^*(y) = \begin{cases} \frac{1-p}{\sqrt{1-y}}, \mathit{for}\ y \le (1-(1-p)^2); \\ 1, \qquad\quad \mathit{for}\ y > (1-(1-p)^2). \end{cases}$$

Proof. Check that the strategy F_p^* of Player 1 equalizes points in spectrum of strategy G_p^* of Player 2, i.e. for all $y \in [0, 1-(1-p)^2]$ value $K_p(F_p^*, y)$ is constant and guarantees to Player 1 the gain $p(1-p)$. Really,

$$\int_0^{1-(1-p)^2} K_p(x,y)dF_p^*(x)$$

$$= (1-p)y - p(1-y)F_p^*(y) + 1/2 \int_y^{1-(1-p)^2} (1-p)(1-x)^{-1/2}dx$$

$$= (1-p)y - (1-p)[(1-y)^{1/2} - (1-y)] + (1-p)(1-y)^{1/2} - (1-p)^2 = p(1-p).$$

It's obvious that for $y > 1-(1-p)^2$ the payoff function decreases. Thus the strategy F_p^* guarantees to Player 1 the gain $p(1-p)$.

Proof for the strategy G_p^* is the similar.

$$\int_0^1 K_p(x,y)dG_p^*(y) = \int_0^{1-(1-p)^2} (1-p)\frac{dG_p^*(y)}{dy}dy + p(1-x)G_p^*(x) -$$

$$\int_x^{1-(1-p)^2} p(1-y)\frac{dG_p^*(y)}{dy}dy = p(1-p).$$

□

Remark 1. Continuous distribution corresponding to the optimal strategy of Player 1 and continuous component of distribution corresponding to the optimal strategy of Player 2 have the same spectrum $(0, 1-(1-p)^2)$ and the similar density proportional to

$$(1-x)^{-3/2}.$$

Remark 2. Changing a scale in results obtained to m we get a formulae for game value

$$V_1(p) = m \cdot p(1-p),$$

following expressions for cumulative density functions: for optimal strategy of Player 1 at state H

$$F_p^*(x) = \begin{cases} \frac{(1-p)(\sqrt{m}-\sqrt{m-x})}{p\sqrt{m-x}}, & \text{for } x \le m(1-(1-p)^2); \\ 1, & \text{for } x > m(1-(1-p)^2). \end{cases}$$

for optimal strategy of Player 2

$$G_p^*(y) = \begin{cases} \frac{(1-p)\sqrt{m}}{\sqrt{m-y}}, & \text{for } y \le m(1-(1-p)^2); \\ 1, & \text{for } y > m(1-(1-p)^2). \end{cases}$$

Densities (similar within a coefficient) of these distributions with the same spectrum $(0, m(1-(1-p)^2))$ are proportional to

$$(m-x)^{-3/2}.$$

3. Model of one-stage bidding with integer bids

Against the model of De Meyer we change a scale: at state H the share price is equal to integer positive number m, at state L the share price is zero.

We consider a model with admissible integer bids proportional to a minimal currency unit. The reasonable bids are only $0, 1, \ldots, m-1$.

The model is reduced to a zero-sum game with lack of information on the side of Player 2. The state space is $S = \{L, H\}$, the action sets of both players are $I = J = \{0, 1, \ldots, m-1\}$.

At state L payoffs that insider receives are given by matrix $A^{L,m}$

$$A^{L,m} = \begin{pmatrix} 0 & 1 & 2 & \ldots m-1 \\ -1 & 0 & 2 & \ldots m-1 \\ -2 & -2 & 0 & \ldots m-1 \\ \vdots & \vdots & \vdots & \ddots \;\; \ldots \\ -m+1 & -m+1 & -m+1 & \ldots \;\; 0 \end{pmatrix}.$$

At state H the payoff matrix $A^{H,m}$ is

$$A^{H,m} = \begin{pmatrix} 0 & -m+1 & -m+2 & \ldots -1 \\ m-1 & 0 & -m+2 & \ldots -1 \\ m-2 & m-2 & 0 & \ldots -1 \\ \vdots & \vdots & \vdots & \ddots \ldots \\ 1 & 1 & 1 & \ldots \;\; 0 \end{pmatrix}$$

Rows of matrix are the bids of Player 1 with numeration starts from zero, column are the bids of Player 2.

It is obvious that at state L insider proposes 0 for any probability p, but at state H insider doesn't use 0. Therefore in the sequel we will be interested in the strategy of Player 1 at state H.

The strategy of Player 2 doesn't depend on the state of nature.

These observations allows to reduce solving of the game $G_1^m(p)$ with incomplete information to solving the game with complete information with payoff matrix

$$A^m(i,j) = \begin{cases} (1-p)j + p(m-i), & \text{for } \; i > j; \\ (1-p)j, & \text{for } \; i = j; \\ (1-p)j + p(-m+j), & \text{for } \; i < j, \end{cases}$$

here $i \in I$ is the bid of insider at state H, $j \in J$ is the bid of uninformed player.

The matrix of gains of insider at state L is reduced to easier view:

$$A^{L,m} = \begin{pmatrix} 0 \; 1 \; 2 \ldots m-1 \end{pmatrix}.$$

The matrix A^m is written down in compact view:

$$A^m(p) = A^{L,m} \cdot (1-p) + A^{H,m} \cdot p.$$

By $V_1^m(p)$ denote the value of the game $G_1^m(p)$ which is expected gain of insider, denote by $v^{H,m}$ and $v^{L,m}$ his gains at states H and L respectively:

$$V_1^m(p) = v^{L,m} \cdot (1-p) + v^{H,m} \cdot p.$$

Logic of constructing optimal strategies of both players according to a probability of choosing high share price is following: when p increases the bids of both players also grow. So for little (close to zero) p bids of both players are minimal. When p increases growth of bids leads to extension of spectrum of selectable strategies.

For p close to $1/2$ Player 2 has maximal uncertainty and hence maximal bids spectrum. Uncertainty is minimal for p close to zero or unity.

It follows from the general theory that the value $V_1^m(p)$ of the game $G_1^m(p)$ is a continuous concave piecewise linear function over $[0, 1]$ with a finite numbers of domains of linearity. Moreover, the optimal strategy of the uninformed Player 2 is constant over linearity domains.

Lets start from p close to zero. Then (due to domination) there is an equilibrium situation in pure strategies, the optimal strategy of Player 2 is proposing zero, ones of Player 1 is proposing 1.

When p grows up starting with certain p_1 a bid 1 of Player 1 becomes to be dominated. Then Player 2 needs to include the bid 1 in his spectrum and similarly Player 1 needs to randomize between bids 1 and 2. Probability p_1 of this strategy changing is the first peak point of function $V_1^m(p)$ and so on.

At each stage uninformed player equalizes insider gains for pure strategies from insider's optimal strategy spectrum. Observing the changes of sets of bids used is optimal strategy by insider one can obtain a peak points of piecewise linear function $V_1^m(p)$.

Similarly for p near 1 one starts from bid $m - 1$ and then add lower bids sequentially. Combination of this approaches (to start from bottom or to start from top) allows to find al peak points of function $V_1^m(p)$ over $[0, 1]$.

4. Analysis of bids used

Consider probabilities from the left part of the interval [0;1]. Let's analyze optimal behavior of players at these probabilities.

Fix p. Denote by x any strategy of insider, by y strategy of Player 2.

Denote the probability (the weight) of action i in this mixed strategy by $x(i)$ and $y(i)$ for Players 1 and 2 respectively.

Definition 1. Set of bids is called *spectrum of strategy* of player if the he use this bids in this mixed strategy with positive probability.

The notation is $\operatorname{Spec} x$, $\operatorname{Spec} y$.

Definition 2. We say that *the strategy y of Player 2 equalizes the subset B of bids* of Player 1 if for all pure strategies of Player 1 from B he receive the same payoff. It means that the following equality holds

$$\sum_j A^m(i,j)y(j) = v \quad \text{for all } i \in B,$$

here v is a common equalization value for all $i \in B$.

In the sequel if we don't mention any set B it means that $B = \operatorname{Spec} x$. The similar definition holds for the case when the strategy x of Player 1 is equalizing for bids of Player 2.

Consider a mixed strategy y, let k_2 is maximal bid in $\operatorname{Spec} y$. Let insider use the maximal bid k_1. Let the pair of strategies (x, y) be optimal. Then spectrums of these strategies are connected by the following way.

Proposition 1. *Let p is not a peak point of function $V_1^m(p)$. Then maximal bid in spectrum of insider's optimal strategy is equal to uninformed player maximal bid or coincides with bid following the maximal uninformed player one, i.e.*

$$k_1 = k_2 \ \textit{ or } \ k_1 = k_2 + 1.$$

Proof. Show first that $k_2 \leq k_1$.

It follows from the structure of matrix $A^m(p)$ that

$$\forall j \in \operatorname{Spec} y \quad j \leq k_1 + 1.$$

Let $j = k_1 + 1 \in \operatorname{Spec} y$. Then the pure strategy $k_1 + 1$ equalizes the strategy of Player 2 and because of uniqueness of extreme optimal strategy $k_1 + 1$ is optimal strategy of Player 2.

If insider uses bids less than or equal to k_1 his gain is equal to

$$A^m(x, k_1 + 1) = -mp + k_1 + 1.$$

But if insider use bid $k_1 + 1$ he will earn

$$A^m(k_1 + 1, k_1 + 1) = -(k_1 + 1)p + k_1 + 1,$$

that is more than $-mp + k + 1$. So it is profitable to insider to deviate from strategy x. It contradicts the optimality of insider's strategy.

Show now that $k_2 > k_1 - 2$. If maximal bid of Player 2 is less than or equal to $k_1 - 2$ then it's not profitable for Player 1 to use the bid k_1 (it is dominated by the bid $k_1 - 1$), but it is impossible because of conditions of the theorem (k_1 is used by insider the in optimal strategy). $\square$

To make notation more clear let's write k instead of k_1; so the maximal bid in spectrum of Player 2 is k or $k - 1$.

Denote the game value $V_1^m(p)$ by $v_k(p)$ if the maximal bid in optimal strategy of insider is k. Also attach index k to the gains of insider at states L and H

$$v_k(p) = v_k^H \cdot p + v_k^L \cdot (1 - p).$$

Consider probabilities p for which maximal bids in spectrums of optimal strategies of both players coincide. Denote the distribution function of weights in strategy y of Player 2 by G_k:

$$G_k(i) = \sum_{j=1}^{i} y(j), \quad G_k(k) = 1,$$

$$y(k) > 0, \quad y(k+1) = \ldots = y(m-1) = 0.$$

Suppose spectrum of insider's strategy is maximal that means it contains all bid from 1 till k without lacunas. As the number of bids should be the same there should exist unusable bid among the bids of Player 2 from 0 till k.

Proposition 2. *Suppose that both players use maximal spectrums in optimal strategies with maximal bid k. Then Player 2 misses either bid 1 or bid 2 in his optimal strategy.*

Proof. It follows from the Shapley-Snow theorem (see Karlin, 1964, Chapter 2) that optimal strategy of Player 2 equalizes the bids of insider. The condition of equalization gives the system of difference equations on G_k:

$$(m-j)G_k(j-1) = (m-j-1)G_k(j+1), \quad j = \overline{1, k-1}. \tag{3}$$

So we obtain $k-1$ equations with k unknown quantities: $G_k(0), G_k(1), \ldots, G_k(k-1)$. ($G_k(k) = 1$ is known.)

Hence *there is one-parametric family of solutions.*

Use a natural monotonicity property of cumulative distribution function: $G_k(j) \leq G_k(j+1)$. Suppose $j > 2$. Let's find a relation between $y(1)$ and $y(2)$.

It follows from (3) that

$$(m-1)y(0) = (m-2)(y(0) + y(1) + y(2))$$

$$y(0) = (m-2)(y(1) + y(2)) \tag{4}$$

Let us suppose, for the sake of definiteness, that k is even. Then using conditions $G_k(k) = 1$ and (3) we move to step 2 and calculate $G_k(0) = y(0)$.

Let's show that for any $y(1)$, $y(2)$ satisfying (4) there exists the equalizing strategy with such $y(1)$, $y(2)$. It is enough to prove that all $y(j) \geq 0$.

Choose $y(1)$. Specify $G_k(1) = y(0) + y(1)$. From the difference equation we find $G_k(3), G_k(5), \ldots, G_k(k-1)$. One can show monotonicity

$$G_k(2j) \leq G_k(2j+1) \leq G_k(2j+2)$$

by induction.

So one can parameterize the family of equalizing strategies of Player 2 by $y(1)$ or $y(2)$ with condition

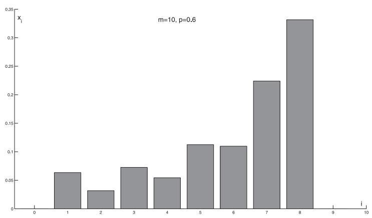
$$y(1) + y(2) = c \geq 0.$$

The optimal strategies corresponds to the extreme points: $y(1) = 0$ and $y(2) = 0$. Therefore the uninformed player misses either bid 1 or bid 2 in his optimal strategy. □

The example of optimal strategies with maximal spectrums is presented on Figure 1.

The spectrum "without lacunas" is not the only possible structure. For some probabilities p spectrums of optimal strategies of both players have lacunas. The possible structures of lacunas are described in the following theorem.

Proposition 3. *The maximal length of lacuna (number of missed bids) in spectrum of optimal strategy of both players is not greater than 1 if players use bids less than $m-1$. The greater lacuna can be only before the bid $m-1$.*



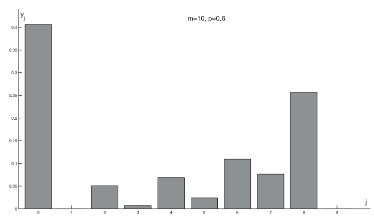


Figure1: Weights of bids of Player 1 and 2 respectively for $m = 10$, $p = 0.6$

Proof. I. Let's verify the statement of the theorem for uninformed player. Proof is by reductio ad absurdum. Suppose Player 2 misses two or more bids. So there exists the bid λ such that

$$y(\lambda - 1) = 0, \quad y(\lambda) = 0, \quad y(\lambda + 1) > 0.$$

Assume that insider uses λ in optimal strategy with positive probability. Then it is profitable for insider to move the weight from bid λ to bid $\lambda - 1$, his additional profit will be

$$x(\lambda)(y(0) + \ldots + y(\lambda - 2)) > 0.$$

It means that the strategy of insider in not optimal and *insider does't post* λ *in optimal strategy*:

$$x(\lambda) = 0.$$

Assume that insider doesn't use bid $\lambda+1$, i.e. $x(\lambda+1) = 0$. Then it is profitable for Player 2 to move the weight from bid $\lambda + 1$ to bid λ, his profit will be

$$y(\lambda + 1)(x(0) + \ldots + x(\lambda - 1)) > 0.$$

It means that the strategy of Player 2 in not optimal and *insider proposes bid* $\lambda + 1$ *with positive probability*:

$$x(\lambda + 1) > 0.$$

It is not profitable for insider to use $\lambda - 1$ instead $\lambda + 1$, that implies

$$A^m(\lambda + 1, y) \geq A^m(\lambda - 1, y).$$

From this inequality we obtain the estimate

$$y(\lambda + 1) \geq \frac{2}{m - (\lambda - 1)} - \frac{(y(\lambda + 2) + \ldots + y(k)) \cdot 2}{m - (\lambda - 1)}. \tag{5}$$

And it is not profitable to deviate from $\lambda + 1$ to $\lambda + 2$ (here we use the fact that $\lambda + 1 < m - 1$, i.e. the lacuna is not before the bid $m - 1$), so

$$A^m(\lambda + 1, y) \geq A^m(\lambda + 2, y).$$

And we obtain an estimate on $y(\lambda + 1)$:

$$y(\lambda + 1) \leq \frac{1}{m - (\lambda + 1)} - \frac{(y(\lambda + 2) + \ldots + y(k))}{m - (\lambda + 1)} - \frac{y(\lambda + 2)(m - (\lambda + 2))}{m - (\lambda + 1)}. \tag{6}$$

Comparing (5) and (6) we get

$$\frac{1}{m-(\lambda+1)} \geq (y(\lambda+2)+\ldots+y(k))\left(\frac{1}{m-(\lambda+1)}-\frac{2}{m-(\lambda-1)}\right)+$$
$$+\frac{2}{m-(\lambda-1)}+\frac{y(\lambda+2)(m-(\lambda+2))}{m-(\lambda+1)}.$$

Using natural conditions one have $y(\lambda+2)+\ldots+y(k) < 1$, $\frac{1}{m-(\lambda+1)} - \frac{2}{m-(\lambda-1)} < 0$:

$$\frac{1}{m-(\lambda+1)} > \frac{1}{m-(\lambda+1)} + \frac{y(\lambda+2)(m-(\lambda+2))}{m-(\lambda+1)},$$

$$0 > y(\lambda+2).$$

Contradiction.
The theorem is proved for Player 2.

II. The proof for insider is similar. □

The example of optimal strategies with lacunas in spectrums is presented on Figure 2.

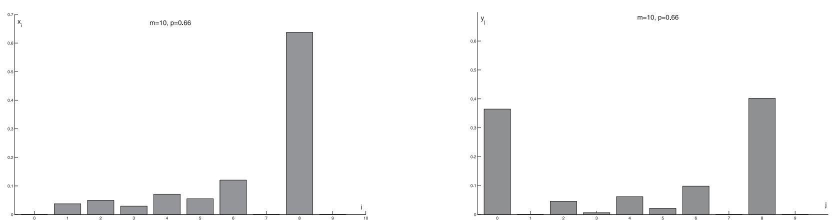


Figure2: Weights of bids of Player 1 and 2 respectively for $m = 10$, $p = 0.66$

5. Procedure of computing of game value

Consider the function $V_1^m(p) = v_k(p)$ of game value. If probability p is not a peak point of function $V_1^m(p)$ then the game value (it coincides with the equalizing value) can be determined the unique way by spectrums of optimal strategies of both players at this probability p. So the following notation takes place

$$v_k(p) = v_k(\mathrm{Spec}\, x, \mathrm{Spec}\, y),$$

k is a maximal bid in $\mathrm{Spec}\, x$.

Consider strategies of players for small probabilities of high share price.

1) Consider p near 0. The game with reduced matrix $A^m(p)$ has an equilibrium situation in pure strategies. The optimal strategy of Player 2 is proposing 0, ones of insider is proposing 1. The vector payoff of insider is

$$v_1^H = m-1,$$

$$v_1^L = 0.$$

The game value is

$$v_1 = (m-1)p.$$

2) Let p increase. Then players pass from pure strategies to mixed strategies. Insider adds the bid 2 to the set of his bids used. Uninformed player adds the bid 1 with such weight that equalize the gain of insider's bids 1 and 2. Therefore the optimal strategy of Player 2 is mixed strategy with weights

$$y = \left(\frac{m-2}{m-1}, \frac{1}{m-1}, 0, 0, \ldots, 0\right),$$

that means $\operatorname{Spec} y = \{0, 1\}$.
The vector payoff of insider and the game value are the following

$$v_2^H = m-2,$$

$$v_2^L = \frac{1}{m-1},$$

$$v_2(p) = (m-2)p + \frac{1}{m-1}(1-p).$$

Hence we find the first peak point p_1 of function $v_k = v(p)$. It's enough to equate the game values from cases 1) and 2):

$$(m-1)p = (m-2)p + \frac{1}{m-1}(1-p),$$

from here

$$1 - p_1 = \frac{m-1}{m}.$$

3) Let p increases a little more. Then Player 2 adds the bid 2 in the spectrum of bids. By property 3 he can't use the bids 1 and 2 in optimal strategy at the same time, so he misses the bid 1. In fact it's the other way to equalize the spectrum of insider $\operatorname{Spec} x = \{1, 2\}$.
The optimal strategy of Player 2 is

$$y = \left(\frac{m-2}{m-1}, 0, \frac{1}{m-1}, 0, \ldots, 0\right).$$

The vector payoff of insider and the game value are the following

$$v_2^H = \frac{(m-2)^2}{m-1},$$

$$v_2^L = \frac{2}{m-1},$$

$$v_2 = \frac{(m-2)^2}{m-1}p + \frac{2}{m-1}(1-p).$$

The second peak point p_2 (when strategy changes from 2) to 3)) admits

$$1 - p_2 = \frac{m-2}{m-1}.$$

Suppose that we know equalizing strategies of Player 2 with all possible spectrums with the maximal bid less than k. Describe in this terms an optimal strategies when insider adds the bid k.

In section 4. we described the properties of spectrums of optimal strategies of both players. As optimal strategies are equalizing we are interested in equalizing strategies with spectrums satisfying properties 1, 2, 3.

Define the operation of *discarding the last* (maximal) *bid* in the strategy of uninformed player by the following way: if the strategy y has a spectrum $\operatorname{Spec} y$ with maximal bid k (for the sake of definiteness we add index k to strategy: y_k) then after discarding the last bid the new strategy (denote it by $\bar{y}_{k-1}$) has a spectrum $\operatorname{Spec} \bar{y}_{k-1} = \operatorname{Spec} y_k \setminus \{k\}$ and weights $\bar{y}_{k-1}(j)$ in strategy $\bar{y}_{k-1}$ are proportional to corresponding weights in initial strategy. Let's introduce the proportionality factor α_k by formulae

$$y_k(j) = \bar{y}_{k-1}(j) \cdot \alpha_k, \quad j = 0, \dots, k-1.$$

Hence

$$y_k(k) = 1 - \alpha_k.$$

5.1. The situation when maximal bid is less than $m - 1$

1) Consider the case when maximal bids in spectrums of optimal strategies of both players coincides and equal to k ($k < m-1$) in detail. Assume that insider uses the bid $k-1$.

Denote by i the pure strategy of player proposing the bid i.

Player 2 equalizes the spectrum of bids of Player 1. The condition of equalization in matrix terms is the following

$$(iA^m(p), (y_k)^t) = v_k, \quad i = 1, \dots, k.$$

(we equalize only the rows according to bids from insider's spectrum) Here v_k is, technically speaking, not a game value, it's an equalization value.

As all rows in matrix $A^{L,m}(p)$ are the same let's rewrite the system above the following way

$$(iA^{H,m}(p), (y_k)^t) = v_k^H, \tag{7}$$

in $A^{H,m}(p)$ only rows and columns corresponding to spectrums elements are used.

Note that it is enough for Player 2 to use bids less than k to equalize insider's spectrum with maximal bid less then k. In other words the k's column in matrix doesn't influence the equalizing a "small" spectrum of insider $\operatorname{Spec} x_k \setminus \{k\}$. So we can discard the last bid in insider's strategy and proceed to consider the strategy $\bar{y}_{k-1}$.

The strategy $\bar{y}_{k-1}$ equalizes the spectrum of insider

$$\operatorname{Spec} x_{k-1} = \operatorname{Spec} x_k \setminus \{k\},$$

with equalization value $v_{k-1} = v_{k-1}^H p + v_{k-1}^L (1-p)$.

The system (7) can be reduced to

$$\begin{cases} v_{k-1}^H \alpha_k + (-m+k) y_k(k) = v_k^H, \\ (m-k)\alpha_k = v_k^H, \\ y_k(k) = 1 - \alpha_k. \end{cases} \tag{8}$$

If one knows v^H_{k-1} for all available combinations of spectrums than he can easily solve the system to obtain the recurrent relation for v^H_k:

$$\alpha_k = \frac{m-k}{v^H_{k-1}},$$

$$y_k(k) = 1 - \frac{m-k}{v^H_{k-1}},$$

$$v^H_k = \frac{(m-k)^2}{v^H_{k-1}},$$

$$v^L_k = (v^L_{k-1} - k)\frac{m-k}{v^H_{k-1}} + k.$$

Obtaining the proportionality factor α_k and the maximal bid weight $y_k(k)$, one constructs the equalizing strategy of Player 2.

The equalization value is

$$v_k(p) = \frac{(m-k)^2}{v^H_{k-1}} \cdot p + \left((v^L_{k-1} - k)\frac{m-k}{v^H_{k-1}} + k \right) \cdot (1-p).$$

2) Here we consider the case when maximal bids in spectrums of optimal strategies of both players equal k, but both players don't use the bid $k-1$.

Forbidding insider to post k and applying the similar reasoning we obtain recurrent relation of the second order:

$$y_k(j) = \bar{y}_{k-2}(j) \cdot \alpha_k, \quad i = 1, \ldots, k-2,$$

$$y_k(k-1) = 0,$$

$$\alpha_k = \frac{m-k}{v^H_{k-2}},$$

$$y_k(k) = 1 - \frac{m-k}{v^H_{k-2}},$$

$$v^H_k = \frac{(m-k)^2}{v^H_{k-2}},$$

$$v^L_k = (v^L_{k-2} - k)\frac{m-k}{v^H_{k-2}} + k,$$

$$v_k(p) = \frac{(m-k)^2}{v^H_{k-2}} \cdot p + \left((v^L_{k-2} - k)\frac{m-k}{v^H_{k-2}} + k \right) \cdot (1-p).$$

3) One more possible case of spectrum structure is the following: maximal bid of insider is k and maximal bid of uninformed player is $k-1$. Moreover there can be lacunas before the maximal bids (their length is not greater than 1). Analysis needed is almost the same as in previous cases.

The mechanism of recursion. *Define the number d by the structure of spectrums of strategies of both players so that the last d bids of uninformed player don't influence the equalizing a "shortened" spectrum of bids of insider*

$$\operatorname{Spec} x \setminus \{k, k-1, \dots, k-(d-1)\}.$$

Let d is minimal ones of all possible.

The equalization value on "short" spectrums of players is v^H_{k-d}. Weights of the last d bids of uninformed player are determined the unique way by conditions of equalizing rows $k, k-1, \dots, k-(d-1)$.

Because of properties 1 - 3 of spectrums the maximal depth of recursion d is not more than 3.

5.2. The situation when players use the bid $m-1$

It follows from property 3 that strategies containing bid $m-1$ have the following structure: they can contain a large lacuna (with length more than 1) before the bid $m-1$, for example from the bid $\tilde{k}$, but there are no lacunas with length more than 1 in interval of bids from 0 to $\tilde{k}$.

Using the notation for weighs of strategies we conclude (for insider) that there exist $\tilde{k} < m-1$ such that

$$x_{m-1}(m-1) > 0, \quad x_{m-1}(m-2) = x_{m-1}(m-3) = \ldots = x_{m-1}(\tilde{k}+1) = 0,$$

$$x_{m-1}(\tilde{k}) > 0, \quad x_{m-1}(j) \geq 0 \quad j = 1, 2, \dots, \tilde{k}-1.$$

Let the maximal bid less than $m-1$ of uninformed player be $\tilde{l}$. Then applying the similar reasoning as in property 1 one can infer the following relation:

$$\tilde{l} = \tilde{k}, \text{ or } \tilde{l} = \tilde{k}-1.$$

Denote the strategy x_{m-1} of insider (with maximal bid $m-1$) by $\tilde{x}_k$. Denote the spectrum of strategy $\tilde{x}_k$ by

$$\operatorname{Spec} \tilde{x}_k = \operatorname{Spec} x_{\tilde{k}} \cup \{m-1\}.$$

We use the same notation for Player 2.

Denote the game value at situation when insider uses the optimal strategy with spectrum $\operatorname{Spec} \tilde{x}_k$ by $\tilde{v}_k$.

Write down the condition of Player 1 optimal strategy spectrum equalization by Player 2:

$$(iA^m(p), (\tilde{y}_l)^t) = \tilde{v}_k, \quad i \in \operatorname{Spec} \tilde{x}_k.$$

It follows from the structure of matrix that the bid $m-1$ of Player 2 does'n influence the equalization of insider's strategy without the last bid $\operatorname{Spec} x_{\tilde{k}}$.

Similarly to the case $k < m-1$ above we introduce the proportionality factor and use the equalization value for strategies with maximal bid equals to $\tilde{k}$. This way we obtain the recurrent formulae on the game value:

$$\tilde{v}^H_k = \frac{1}{v^H_{\tilde{k}}},$$

$$\tilde{v}^L_k = (v^L_{\tilde{k}} - m + 1) \cdot \frac{1}{v^H_{\tilde{k}}} + m - 1,$$

$$\tilde{v}_k = \frac{1}{v^H_{\tilde{k}}} \cdot p + \left((v^L_{\tilde{k}} - m + 1) \cdot \frac{1}{v^H_{\tilde{k}}} + m - 1 \right) \cdot (1 - p).$$

Consider a special case when $\tilde{k}$ doesn't exist that means that the optimal strategies of both players are pure strategies – the bid $m-1$. It occurs when probability p is near 1 namely $p \in [\frac{m-1}{m}, 1]$.

When recursion goes to step 2 we use the initial conditions (were found above). As a result we obtain the weights for case when players use strategies with maximal spectrums. As for $k = 2$ where one has two families of equalizing strategies of Player 2 at each step of recursion we get two families of equalizing strategies with maximal spectrums.

The number of possible combinations of spectrums of equalizing strategies is limited by properties 1, 2, 3. Therefore the number of equalizing strategies for each probability p (and so for domains of linearity of function $V_1^m(p)$) is limited. Choose as an optimal strategy of Player 2 the one that gives to insider minimal guaranteed benefit. The corresponding equalizing value is a game value for this p.

We have in mind in this case the following. As the same spectrum of insider can be equalized not uniquely by Player 2, he must choose the strategy that gives the minimal benefit to insider. In this case insider chooses the strategy with such spectrum to make this minimum the biggest from all possible minimums. Therefore the choice is realized on base of min-max theorem applying to the "small" set of equalizing strategies.

6. Peak points of function $V_1^m(p)$

It the property 2 it was established that if players use maximal possible bids then Player 2 can equalize the spectrum of Player 1 in two ways: using all bids excluding the bid 1 or 2. Denote these equalizing strategies of Player 2 by y_k^1 (if the bid 1 is used) y_k^2 (if the bid 2 is used), k is a maximal bid in spectrums of both players.

We obtain two families of equalizing strategies. Denote the probabilities when the strategies y_k^1 and y_k^2 interchange (for $k < m-1$) by p_k, $\{p_k\} = P^m$. Denote the probabilities when the strategies $\tilde{y}_k^1$ and $\tilde{y}_k^2$ interchange by q_k, $\{q_k\} = Q^m$.

One can explicitly compute this probabilities. In section 5. were compute p_1 and p_2. For $k > 2$ the recurrent formula holds:

$$1 - p_1 = \frac{m-1}{m}, \quad 1 - p_2 = \frac{m-2}{m-1}, \quad 1 - p_k = (1 - p_{k-2}) \frac{m-k}{m-k+1}.$$

The similar formulae for the set Q_m:

$$1 - q_1 = \frac{1}{m}, \quad 1 - q_2 = \frac{1}{m-1}, \quad 1 - q_k = (1 - p_{k-2}) \frac{1}{m-k+1}.$$

These formulae determine the same families of probabilities which were obtained in the work of Kreps, 2009a. These families have properties established in the mentioned paper. Here we adduce these properties.

Theorem 2. *For $p \in P^m \bigcup Q^m$ the value of the game with admissible integer bids coincides with the value of the game with arbitrary bids (De Meyer's model):*

$$V_1^m(p) = m \cdot p \cdot (1 - p) \quad \textit{for all } p \in P^m \bigcup Q^m.$$

Theorem 3. *The set $P^m \bigcup Q^m$ becomes everywhere dense over $[0,1]$ as $m \to \infty$. One has as corollary:*

$$\lim_{m\to\infty} V_1^m(p)/m = p(1-p).$$

So the probabilities from the set $P^m \bigcup Q^m$ have a marvelous property that at these points there are no difference for players to play the game with real or integer bids. At other points p insider prefers to play the game with arbitrary bids because in this game he has the greater freedom of action that guarantees him the greater benefit.

7. On the optimal strategy of insider

Consider the strategy x_k of insider (with maximal bid k in it's spectrum). Denote the distribution function of weights in strategy x_k by F_k:

$$F_k(i) = \sum_{j=1}^{i} x_k(j), \quad F_k(k) = 1,$$

$$y(k) > 0, \quad y(k+1) = \ldots = y(m-1) = 0.$$

In section 5. for fixed p we found the optimal strategy y of uninformed player (so $\operatorname{Spec} y$ was found too) and the game value $v_k(p)$. Withal the spectrum $\operatorname{Spec} x$ of optimal strategy x of insider at state H was found.

Strategy of insider equalizes active bids of Player 2. If $j-1$ and j are active bids of Player 2 then the condition of equalization is in the form of second-order difference equation:

$$F_k(j-2) - \frac{m-j}{m-j+1} F_k(j) + \frac{1}{m-j+1} \cdot \frac{1-p}{p} = 0.$$

If the bids $j-2$ and j are active, but Player 2 misses bid $j-1$, then one should use the condition of "gluing"

$$(x_k \cdot A^m(p), (j)^t) = (x_k \cdot A^m(p), (j-2)^t) \quad (= v_k(p)).$$

From this it follows the third-order difference equation on the cumulative distribution function:

$$F_k(j-3)(m-j) + (F_k(j-2) - F_k(j+1))(m-j+1) - F_k(j)(m-j) + 2\frac{1-p}{p} = 0.$$

In the case of $k = m-1$ the similar condition is given for the bids $m-1$ and $\tilde{k}$.

Another conditions on F_k follows from the analysis of $\operatorname{Spec} x$. If the bid i is not in $\operatorname{Spec} x$. then this gives the following simple condition on F_k:

$$F_k(i) = F_k(i-1).$$

So we obtain a sufficient number of independent linear equations to uniquely determine the distributions of weights in optimal strategy of insider. Solution of the system of equations gives the optimal strategy of insider.

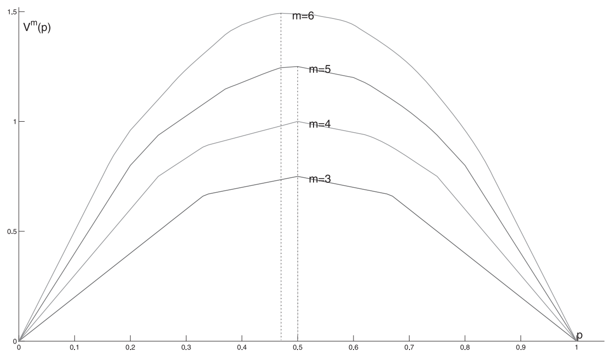


Figure3: The gains of insider in one-stage game for m from 3 till 6

8. The results of computer simulation

Here our aim is to investigate the game value which was described in section 5. by means of computer modeling. The function $V_1^m(p)$ for $m = 3, 4, 5$ and 6 is presented on a Figure 3.

By heuristic analysis of the game it seems to be natural that Player 2 has maximal uncertainty for $p = 1/2$. It means that insider gets the maximal gain at $p = 1/2$. Moreover, the result of continuous model of De Meyer and Saley confirms this idea. As it was shown in section **??** the value of continuous game is a quadratic function with it's maximum at the point $p = 1/2$:

$$V^m(p) = m \cdot p\,(1 - p).$$

In discrete model for small m ($m \leq 5$) the maximum of the game value function is observed at the point $p = 1/2$. But starting with $m = 6$ the maximum of $V_1^m(p)$ shifts a little (it is shown dashed on Figure 3). It means that *the maximum of uncertainty is not obliged to be at* $p = 1/2$. It's the first counterintuitive property of the discrete model.

The second counterintuitive property is *observed dissymmetry of the dame value with respect to the probability* 1/2. It means that in the general case the game values at points p and $1 - p$ aren't equal.

One more nontrivial effect of the model is *possibility of lacunas in spectrums of optimal strategies*. In the continuous model there are no any lacunas in the interval between minimal and maximal bids used.

9. Conclusion

The obtained results demonstrate that the discrete model possesses a number of specific characteristics that distinguish it from the continuous models. Hence the methods of analysis of discrete models are fundamentally different. The developed recurrent approach demonstrates complexity and originality of solving this class of models.

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Locally Optimizing Strategies for Approaching the Furthest Evader

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Abstract We describe a method for constructing feedback strategies based on minimizing/maximizing state evaluation functions with use of steepest descent/ascent conditions. For a specific kinematics, not all control variables may be presented implicitly in the corresponding optimality conditions and some additional local conditions are to be invoked to design strategies for these controls. We apply the general technics to evaluate a chance for the pursuer P to approach the real target that uses decoys by a kill radius r. Assumed that P cannot classify the real and false targets. Therefore, P tries to come close to the furthest evader, and thereby to guarantee the capture of all targets including the real one. We setup two-person zero-sum differential games of degree with perfect information of the pursuing, P, and several identical evading, $E_1, \ldots, E_N$, agents. The P's goal is to approach the furthest of $E_1, \ldots, E_N$ as closely as possible. Euclidean distances to the furthest evader at the current state or their smooth approximations are used as evaluation functions. For an agent with simple motion, the method allows to specify the strategy for heading angle completely. For an agent that drives a Dubins or Reeds-Shepp car, first we define his targeted trajectory as one that corresponds to the game where all agents has simple motions and apply the locally optimal strategies for heading angles. Then, to design strategies for angular and ordinary velocities, local conditions under which the resulting trajectories approximate the targeted ones are invoked. Two numerical examples of the pursuit simulation when one or two decoys launched at the initial instant are given.

Keywords: Conservative pursuit strategy, Lyapunov-type function, steepest descent/ascent condition, smooth approximation for min/max, Dubins car, Reeds-Shepp car, decoy.

1. Introduction

In spite of tremendous progress made through its a relatively short history, theory of differential games doesn't provide direct methods and tools for solving concrete games. In this paper, we describe a method for minimax optimization of terminal outcomes. Minimizing/maximizing feedback strategies are constructed with use of steepest descent/ascent conditions for the corresponding state evaluation functions; see, e.g., (Sticht et al., 1975 ; Shevchenko, 2008; Stipanović et al., 2009). For a specific kinematics, not all control variables may be presented implicitly in the corresponding optimality conditions. Therefore, some additional local conditions are needed to be invoked to design strategies for the remaining controls.

To demonstrate the general technics, we analyze models of conflict situations where an agile pursuer in the plane strives to approach a slower target by a given distance. Both sides rely on sensors that perfectly measure positions without time delays. To avoid an inevitable capture, the target changes the information conditions by launching one or several decoys (false targets) that introduce "false positive" errors for sensing system of the pursuer (Pang, 2007). Thus, after firing the decoys, the pursuer faces several identical targets instead of one. It makes his task much more complicated compared to the original one that could be accomplished just by following the real target's trajectory.

Very little work has been done on pursuit-evasion with decoys. Lewin (1973) considered decoys only with a finite duration of functioning. Breakwell et al. (1979), Abramyants et al. (1980), Shevchenko (1982) assumed that only one decoy is launched and the pursuer P can classify a target if it is closely enough. The payoff equals the total time spent for a successive pursuit in the worst case when P first approaches the false target for its classification and then capture the real one. Solutions for several other related simple games are given, e.g., in (Shevchenko, 1997; Shevchenko, 2004a; Shevchenko, 2004b; Shevchenko, 2009). Some warfare and combat applications are described, e.g., by Armo (2000), Cho et al. (2000), Pang (2007).

In this paper, to generate a pursuit strategy and evaluate a chance for the pursuer to succeed with approaching the real target by a kill radius r, we setup two-person zero-sum differential games of degree with perfect information of the pursuing, P, and several identical evading, $E_1, \dots, E_N$, agents. The P's goal is to approach the furthest of $E_1, \dots, E_N$ as closer as possible. To construct conservative pursuit strategies and to determine instants of the pursuit termination, Euclidean distances to the furthest of $E_1, \dots, E_N$ at the current state or their smooth approximations (Shevchenko, 2008; Stipanović et al., 2009) are used as evaluation functions. Plane kinematics of the parties is described by some transition equations for wheeled robots (LaValle, 2006; Patsko and Turova, 2009). For an agent with simple motion, steepest descent/ascent conditions for these functions allow to specify his strategy for heading angle completely. For an agents that drive Dubins or Reeds-Shepp cars, the targeted trajectory correspond to the strategies for heading angles in the game where all agents have simple motions. To design strategies for angular and ordinary velocities, local conditions are invoked that make the resulting trajectories as closely as possible to the targeted ones. Two numerical examples of the pursuit simulation when one or two decoys launched at the initial instant are given.

2. Common Optimality Conditions

Let $z_P(t) \in \mathbb{R}^{n_P}$ and $z_e(t) \in \mathbb{R}^{n_e}$ obey the separable equations

$$\begin{aligned} \dot{z}_P(t) &= f_P(z_P(t), u_P(t)), \quad z_P(0) = z_P^0, \\ \dot{z}_e(t) &= f_e(z_e(t), u_e(t)), \quad z_e(0) = z_e^0, \end{aligned} \tag{1}$$

where $t \geq 0$, $u_P(t) \in \mathsf{U}_P \subset \mathbb{R}^{m_P}$, $u_e(t) \in \mathsf{U}_e \subset \mathbb{R}^{m_e}$, U_P and U_e are compact sets, $f_P : \mathbb{R}^{n_P} \times \mathsf{U}_P \to \mathbb{R}^{n_P}$ and $f_e : \mathbb{R}^{n_e} \times \mathsf{U}_e \to \mathbb{R}^{n_e}$, z_P^0 and z_e^0 are the initial positions, $e \in E = \{E_1, \dots, E_N\}$. Suppose that $\mathrm{co}\{f_k(z_k, \mathsf{u}_k) : u_k \in \mathsf{U}_k\} = f_k(z_k, \mathsf{U}_k)$, $z_k \in \mathbb{R}^{n_k}$, $k \in K = \{P\} \cup E$.

Let $M = n_P + n_{E_1} + \ldots + n_{E_N}$, $z = (z_P, z_E) \in Z = \mathbb{R}^M$, and

$$\dot{z}(t) = f(z(t), u_P(t), u_E(t)),\ z(0) = z^0, \tag{2}$$

where $u_E = (u_{E_1}, \ldots, u_{E_N})$, $f_E(z_E, u_E) = (f_{E_1}(z_{E_1}, u_{E_1}), \ldots, f_{E_N}(z_{E_N}, u_{E_N}))$, $z^0 = (z^0_P, z^0_E)$, $f(z, u_P, u_E) = (f_P(z_E, u_P), f_E(z_E, u_E))$. We assume that f is jointly continuous and locally Lipschitz with respect to z, and satisfies the extendability condition; see, e.g., (Subbotin and Chentsov, 1981).

Let $\mathcal{K} : Z \to \mathbb{R}^+$ be a directionally differentiable function that evaluates a given state, and P/E strive to get a lowest/highest value of $\mathcal{K}$ along trajectories of (2) by a given or chosen by P instant $t = \tau \geq 0$. We define locally optimizing strategies $U^l_P \div u^l_P(z) : Z \to \mathsf{U}_P$ and $U^l_e \div u^l_e(z) : Z \to \mathsf{U}_e$ as the functions that meet the following steepest descent/ascent conditions,

$$\begin{aligned} f_P(z_P, u^l_P(z)) &\in \operatorname{Arg} \min_{\upsilon_P \in \operatorname{co} f_P(z_P, \mathsf{U}_P)} \partial_{\upsilon_P} \mathcal{K}(z), \\ f_e(z_e, u^l_e(z)) &\in \operatorname{Arg} \max_{\upsilon_e \in \operatorname{co} f_e(z_e, \mathsf{U}_e)} \partial_{\upsilon_e} \mathcal{K}(z),\ e \in E. \end{aligned} \tag{3}$$

At the points of differentiability,[1] condition (3) may be rewritten as

$$\begin{aligned} u^l_P(z) &\in \operatorname{Arg} \min_{u_P \in \mathsf{U}_P} \frac{\partial}{\partial z_P} \mathcal{K}(z) \cdot f_P(z_P, u_P), \\ u^l_e(z) &\in \operatorname{Arg} \max_{u_e \in \mathsf{U}_e} \frac{\partial}{\partial z_e} \mathcal{K}(z) \cdot f_e(z_e, u_e),\ e \in E; \end{aligned} \tag{4}$$

see, e.g., (Sticht et al., 1975 ; Shevchenko, 2008; Stipanović et al., 2009)

For a given duration $\tau > 0$, initial state $z^0 \in Z$, partition Δ of $[0, \tau]$, $\Delta = \{t_0, t_1, \ldots t_n\}$, $t_0 = 0$, $t_n = \tau$, $\delta t_i = t_{i+1} - t_i$, $i = 0, 1, \ldots, n-1$, and pursuit strategy U^l_P, consider a differential inclusion

$$\dot{z}(t) \in \operatorname{co} f(z(t_i), U^l_P(z(t_i)), \mathsf{U}_E),\ z(0) = z^0, \tag{5}$$

for $t_i \leq t < t_{i+1}$ where $\mathsf{U}_E = (\mathsf{U}_{E_1}, \ldots, \mathsf{U}_{E_N})$. Let $Z_P(z^0, \mathcal{U}^l_P, \Delta)$ be a set of continuous functions $[0, \tau] \to Z$ that are absolutely continuous and meet (5) for almost all $t \in (0, \tau)$; see, e.g., (Subbotin and Chentsov, 1981). Let us evaluate $\mathcal{K}$ by the instant τ. Since $\mathcal{K}$ is directionally differentiable,

$$\mathcal{K}(z(t_{i+1})) - \mathcal{K}(z(t_i)) = \partial_{\upsilon_i} \mathcal{K}(z(t_i)) \delta t_i + o(\delta t_i) \tag{6}$$

where $t_{i+1} = t_i + \delta t_i$, $z(t_{i+1}) = z(t_i) + \upsilon_i \delta t_i$, $\upsilon_i \in f(z(t_i), U_P(z(t_i)), \mathsf{U}_E)$, $i = 0, 1, \ldots, n-1$. From (6) we obtain that

$$\mathcal{K}(z(t_n)) - \mathcal{K}(z(t_0)) = \sum_{i=0}^{n-1} \partial_{\upsilon_i} \mathcal{K}(z(t_i)) \delta t_i + o(|\Delta|), \tag{7}$$

$|\Delta| = \max \delta t_i$, and $\partial_{\upsilon_i} \mathcal{K}(z(t_i)) \leq \partial_{\upsilon^l_i} \mathcal{K}(z(t_i))$ where $\upsilon^l_i = f(z(t_i), u^l_P(z(t_i)), u^l_E(z(t_i)))$, $i = 0, 1, \ldots, n-1$. Thus we have

$$\mathcal{K}(z(t_n)) - \mathcal{K}(z(t_0)) \leq \sum_{i=0}^{n-1} \partial_{\upsilon^l_i} \mathcal{K}(z(t_i)) \delta t_i + o(|\Delta|). \tag{8}$$

[1] For example, $\mathcal{K}$ is differentiable at almost all points of Z if it is uniformly Lipschitz continuous in every open set (Friedman, 1999).

Let $U_P^l \div u_P^l(z)$ and $U_e^l \div u_e^l(z)$ be strategies uniquely defined with use of (3), $k^l(t) = \partial_{\dot{z}^l(t)}\mathcal{K}(z^l(t))$, where

$$\dot{z}^l(t) = f(z^l(t), u_P^l(z^l(t)), u_E^l(z^l(t))),\ z^l(0) = z^0.$$

Theorem 1. *If k^l is integrable on $[0,\tau]$ and $\int_0^\tau k^l(t)dt < 0$ then $\mathcal{K}(z^l(\tau)) < \mathcal{K}(z^0)$.*

Thus, under the assumptions of the theorem, P guarantees a decrease in the initial value of $\mathcal{K}$ by $t = \tau$ when the agents use U_P^l and U_E^l. When τ is not given, P proceeds until the first instant when k^l changes its sign from minus to plus if the assumptions of the theorem are met, or terminates the game at the initial instant otherwise.

3. Distance to Furthest Evader

Let $\rho_e(z)$ be Euclidean distance from e to P at the state $z \in Z$, $e \in E$, and

$$\mathcal{K}^\infty(z) = \max\nolimits_{e\in E} \rho_e(z), \tag{9}$$

be the valuation function. If $\pi_k(z)$, $k \in K$, are Cartesian coordinates of the k-th agent at the state z then $\rho_e(z) = ||\pi_e(z) - \pi_P(z)||$. Since max and ρ_e are convex, $\mathcal{K}^\infty$ is directionally differentiable. It is known (see, e.g., (Subbotin and Chentsov, 1981; Dem'yanov and Vasilev, 1985)) that for all $z, \upsilon \in Z$,

$$\partial_\upsilon \mathcal{K}^\infty(z) = \max\nolimits_{e\in E_0(z)} \partial_\upsilon \rho_e(z), \tag{10}$$

where $E_0(z) = \{e \in E : \rho_e(z) = \mathcal{K}^\infty(z)\}$,

$$\partial_\upsilon \rho_e(z) = \begin{cases} \dfrac{\partial}{\partial z}\rho_e(z)\cdot \upsilon \text{ if } \rho_e(z) \neq 0, \\ ||\upsilon|| \text{ otherwise.} \end{cases} \tag{11}$$

3.1. Smooth Upper Approximations for $\mathcal{K}^\infty$

Conditions (3) are replaced by (4) if a smooth upper approximation is used for the valuation function. Describe some approximations for $\mathcal{K}^\infty$ and their properties; see, e.g., (Shevchenko, 2008; Stipanović et al., 2009).

Let $m_2^\xi(r_1, r_2) = (r_1^\xi + r_2^\xi)^{1/\xi}$, $\xi, r_1, r_2 \in \mathbb{R}^+$. It is known that

$$m_2^\xi(r_1, r_2) > \max(r_1, r_2) \text{ if } r_1 \neq r_2,$$
$$m_2^\xi(r, r) = 2^{1/\xi} r > \max(r, r) = r,$$

and

$$\lim_{\xi\to+\infty} m_2^\xi(r_1, r_2) = \max(r_1, r_2), \quad \xi, r_1, r_2, r \in \mathbb{R}^+. \tag{12}$$

Figure 1 shows projections of $\max(r_1, r_2)$, $m_2^\xi(r_1, r_2)$ and $M_2^\xi(r_1, r_2)$ for a fixed value of r_2 where $M_2^\xi = (r_1^{\xi+1} + r_2^{\xi+1})/(r_1^\xi + r_2^\xi)$, $\xi, r_1, r_2 \in \mathbb{R}^+$, approximates max from below (Shevchenko, 2009), $\xi = 50$.

Lemma 1. *The partial derivative*

$$\frac{\partial}{\partial r_i} m_2^\xi(r_1, r_2) = \left(r_i^\xi/(r_1^\xi + r_2^\xi)\right)^{1-1/\xi}, \quad \xi, r_1, r_2 \in \mathbb{R}^+,\ i = 1, 2,$$

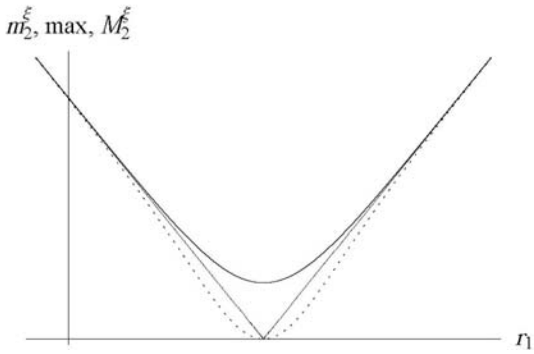

Figure1: Projections of m_2^ξ, max (thin) and M_2^ξ (dotted)

approximates the corresponding partial derivative of $\max(r_1, r_2)$ *at the points where* $r_1 \neq r_2$ *such that*

$$\lim_{\xi \to +\infty} \frac{\partial}{\partial r_i} m_2^\xi(r_1, r_2) = \begin{cases} 1 & \text{if } r_i = \max(r_1, r_2), \\ 0 & \text{if } r_i < \max(r_1, r_2), \end{cases} \tag{13}$$

and

$$\frac{\partial}{\partial r_i} m_2^\xi(r, r) = (1/2)^{1-1/\xi},\ \xi, r_1, r_2, r \in \mathbb{R}^+,\ i = 1, 2.$$

□

An approximation for $\max(r_1, \ldots, r_N)$ may be constructed as

$$m_N^\xi(r_1, r_2, \ldots, r_N) = m_2^\xi(m_{N-1}^\xi(r_1, r_2, \ldots, r_{N-1}), r_N) = \Big(\sum_{k \in K_N} r_k^\xi \Big)^{1/\xi}. \tag{14}$$

It turns out that $m_N^\xi(r_1, \ldots, r_N) > \max(r_1, \ldots, r_N)$ and

$$\lim_{\xi \to +\infty} m_N^\xi(r_1, \ldots, r_N) = \max(r_1, \ldots, r_N), \quad \xi, r_1, \ldots, r_N, r \in \mathbb{R}^+, \tag{15}$$

if $r_i \neq r_j$, $\forall i \neq j$, $i, j = 1, \ldots, N$. Moreover,

$$\lim_{\xi \to +\infty} m_N^\xi(r_1, \ldots, r_N) = (1/k)^{1/\xi}\, r,$$

if there exist exactly $k \geq 2$ arguments $r_{l_1}, \ldots, r_{l_k}$ such that $r_{l_1} = \ldots = r_{l_k} = r = \max(r_1, \ldots, r_N)$, $r_j < \max(r_1, \ldots, r_N), \forall j \neq l_q, q = 1, \ldots, k$.

Theorem 2. *The partial derivative*

$$\frac{\partial}{\partial r_i} m_N^\xi(r_1, \ldots, r_N) = \Big(r_i^\xi / (r_1^\xi + \ldots + r_N^\xi) \Big)^{1-1/\xi}, \quad \xi, r_1, \ldots, r_N \in \mathbb{R}^+,$$

approximates the corresponding partial derivative of $\max(r_1, \ldots, r_N)$ *such that*

$$\lim_{\xi \to +\infty} \frac{\partial}{\partial r_i} m_N^\xi(r_1, \ldots, r_N) = \begin{cases} 1 & \text{if } r_i = \max(r_1, \ldots, r_N), \\ 0 & \text{if } r_i < \max(r_1, \ldots, r_N), \end{cases} \tag{16}$$

if $r_i \neq r_j, \forall i \neq j,\ i,j = 1, \ldots, N,$ *and*

$$\lim_{\xi \to +\infty} \frac{\partial}{\partial r_i} m_N^\xi(r_1, \ldots, r_N) = (1/k)^{1-1/\xi},\ \xi, r_1, \ldots, r_N, r \in \mathbb{R}^+,$$

if $r_i = r = \max(r_1, \ldots, r_N)$ *and there exist exactly* $k \geq 2$ *arguments such that* $r_{i_1} = \ldots = r_{i_k} = r$ *and* $r_j < \max(r_1, \ldots, r_N), j \neq i_q, q = 1, \ldots, k.$

□

Smooth approximations for $\mathcal{K}^\infty$ may be represented as the superpositions of m_N^ξ and $\rho_e(z)$, $e \in E$,

$$\mathcal{K}^\xi(z) = m_N^\xi(\rho_{E_1}(z), \ldots, \rho_{E_N}(z)),\ \xi \in \mathbb{R}^+, z \in Z, N \geq 2.$$

3.2. Specifications for Different Kinematics

First, we demonstrate direct applicability of the described conditions for constructing locally optimizing pursuit strategies for agents that have simple motion. In the case when an agent drives Dubins or Reeds-Shepp car, the control variables for heading angles are not presented in (3) and (4) with $\mathcal{K} = \mathcal{K}^\xi$ implicitly. Accordingly, we use the designed strategies and supplement the common conditions by some additional local ones for the heading angles; see, e.g., (Stipanović et al., 2009).

Simple Motion Let $s_k = (x_k, y_k)$ and φ_k be the coordinate vector and heading angle of the k-th agent, $k \in K$. For the k-th agent with simple motion,

$$\dot{s}_k = \mu_k \epsilon(u_k), \quad s_k(0) = s_k^0, \tag{17}$$

where μ_k is the constant speed, u_k is the control for heading angle, $u_k \in \mathsf{U}_k = \{u : 0 \leq u < 2\pi\}$, $\epsilon(\alpha) = (\cos\alpha, \sin\alpha)$, $k \in K$. If $\xi < \infty$ and $\rho_e(s) = ||s_e - s_P|| \neq 0$, $s = (s_P, s_{E_1}, \ldots, s_{E_N})$, from (3) we have $U_P^\xi \div \varphi_P^\xi(s)$ where

$$\epsilon(\varphi_P^\xi(s)) = -\sum_{e \in E} \frac{\partial}{\partial s_P} \mathcal{K}^\xi(s) / || \sum_{e \in E} \frac{\partial}{\partial s_P} \mathcal{K}^\xi(s)||, \tag{18}$$

if $|| \sum_{e \in E} \dfrac{\partial}{\partial s_P} \mathcal{K}^\xi(s)|| \neq 0$. Also, $U_e^\xi \div \varphi_e^\xi(s)$ where

$$\epsilon(\varphi_e^\xi(s)) = \frac{\partial}{\partial s_e} \mathcal{K}^\xi(s) / || \frac{\partial}{\partial s_e} \mathcal{K}^\xi(s)|| = \frac{s_e - s_P}{||s_e - s_P||}, \tag{19}$$

if $|| \dfrac{\partial}{\partial s_e} \mathcal{K}^\xi(s)|| \neq 0$, $e \in E, \xi \in R^+, s \in S = \mathbb{R}^{2N+2}$.

If $\xi = \infty$, $\rho_e(s) \neq 0$ and there is just one furthest evader E_{i_0} at the state s, (18) and (19) work only for P and E_{i_0} with $\epsilon(\varphi_P^\xi(s)) = \epsilon(\varphi_{E_{i_0}}^\xi(s)) = (s_{E_{i_0}} - s_P)/||s_{E_{i_0}} - s_P||$ and

$$\lim_{\xi \to +\infty} \varphi_P^\xi(s) = \varphi_P^\infty(s),\ \lim_{\xi \to +\infty} \varphi_{E_{i_0}}^\xi(s) = \varphi_{E_{i_0}}^\infty(s),\ s \in S. \tag{20}$$

Dubins Car. If the k-th agent drives a Dubins car,

$$\begin{aligned} \dot{s}_k &= \mu_k \epsilon(\varphi_k), \quad s_k(0) = s_k^0, \\ \dot{\varphi}_k &= w_k, \quad \varphi_k(0) = \varphi_k^0, \end{aligned} \tag{21}$$

the heading angle φ_k is the integral of the angular velocity φ_k which is the only control variable, $w_k \in \mathsf{W}_k = \{w : |w| \leq \nu_k\}$, $k \in K$.

Condition (3) doesn't include w_k explicitly. To choose a strategy for w_k with use of the known locally optimizing strategies U_k^ξ described by (18) and (19), consider the piecewise constant controls and corresponding trajectories.

Let Δ be a partition of $[0, \tau]$ and the targeted direction for the k-th agent at the state $z = (s, \ldots, \varphi_k, \ldots)$ be determined by the angle $\varphi_k^\xi(s)$. Then, the part of the targeted trajectory for $t \in [t_i, t_{i+1})$ is approximately described as

$$s_k(t) = s_k(t_i) + \mu_k \epsilon(\varphi_k^\xi(s(t_i)))\delta t_i. \tag{22}$$

On the other hand, the same part of the approximate trajectory for a piecewise constant control $w_k = w_k(t_{i-1})$ for $t \in [t_{i-1}, t_i)$ is determined by

$$s_k(t) = s_k(t_i) + \mu_k \epsilon(\varphi_k(t_{i-1}) + w_k(t_{i-1})\delta t_{i-1})\delta t_i. \tag{23}$$

Let $w_k(t_{i-1})$ be chosen to make (23) as closely to (22) as possible with use of the condition

$$w_k(t_{i-1}) \in \operatorname{Arg} \min_{w \in \mathsf{W}_k} |\mu_k \epsilon(\varphi_k(t_{i-1}) + w\delta t_{i-1}) - \mu_k \epsilon(\varphi_k^\xi(s(t_i)))|, \tag{24}$$

or for the corresponding heading angles,

$$w_k(t_{i-1}) \in \operatorname{Arg} \min_{w \in \mathsf{W}_k} |\varphi_k(t_{i-1}) + w\delta t_{i-1} - \varphi_k^\xi(s(t_i))|. \tag{25}$$

It means that

$$w_k(t_{i-1}) = \begin{cases} (\varphi_k^\xi(s(t_i)) - \varphi_k(t_{i-1}))/\delta t_{i-1} \\ \qquad \mathit{if}\ |(\varphi_k^\xi(s(t_i)) - \varphi_k(t_{i-1}))/\delta t_{i-1}| \leq \nu_k \\ -\nu_k \operatorname{sgn}(\varphi_k(t_{i-1}) - \varphi_k^\xi(s(t_i)))\ \mathit{otherwise}. \end{cases} \tag{26}$$

From (26), with some abuse of notation we have (compare to (Stipanović et al., 2009)),

$$w_k^\xi(s, \varphi_k) = \begin{cases} \partial_{\omega^\xi(s)} \varphi_k^\xi(s)\ \mathit{if}\ |\partial_{\omega^\xi(s)} \varphi_k^\xi(s)| \leq \nu_k \\ -\nu_k \operatorname{sgn}(\varphi_k - \varphi_k^\xi(s))\ \mathit{otherwise}, \end{cases} \tag{27}$$

where $\omega^\xi(s) = (\mu_P \epsilon(\varphi_P^\xi(s)), \mu_{E_1} \epsilon(\varphi_{E_1}^\xi(s)), \ldots, \mu_{E_N} \epsilon(\varphi_{E_N}^\xi(s)))$. Thus, at the states where the current heading angle allows to move at the locally optimizing direction for simple motion, the agent does so. Otherwise, he tries to compensate the difference between the actual and targeted heading angles.

Reeds-Shepp Car. A Reeds-Shepp car can change its velocity instantaneously and, e.g., move in reverse. Corresponding equations for the k-th agent are written as

$$\begin{aligned} \dot{s}_k &= v_k \epsilon(\varphi_k), \quad s_k(0) = s_k^0, \\ \dot{\varphi}_k &= w_k, \quad \varphi_k(0) = \varphi_k^0, \end{aligned} \tag{28}$$

where v_k is the velocity, $v_k \in \mathsf{V}_k = \{v : |v| \leq \mu_k\}$, $w_k \in \mathsf{W}_k$.

If $\xi < \infty$ and $\rho_e(s) \neq 0$, the common local optimizing conditions lead to the strategies $V_k \div v_k(s, \varphi_k)$, $k \in K$,

$$\begin{aligned} v_P^\xi(s, \varphi_P) &= -\mu_k \operatorname{sgn} \cos(\varphi_P - \varphi_P^\xi(s)), \\ v_e^\xi(s, \varphi_e) &= \mu_e \operatorname{sgn} \cos(\varphi_e - \varphi_e^\xi(s)),\ e \in E. \end{aligned} \tag{29}$$

For w_k the agent may apply the strategy $W_k^\xi \div w_k^\xi(s, \varphi_k)$, $k \in K$.

With use of the method, it is easy to construct the locally optimal strategies for agents in the games with homicidal chauffeur dynamics including the acoustic variant when $w_k \in [-\hat{\mu}_k, \hat{\mu}_k]$, $\hat{\mu}_k(s) = \mu_k \min(1, \rho_{E_k}(s)/\rho*)$, $\rho* > 0$, and other generalizations when $w_k \in [\check{\mu}_k, \mu_k]$, $-\mu_k \leq \check{\mu}_k \leq \mu_k$ (Patsko and Turova, 2009).

Numeric Simulations. Figures 2 and 3 show trajectories of the agents, K^ξ and its time derivative as well as controls and heading angles as functions of i. For all examples, $s_P^0 = (0, 0)$, P has simple motion, $\mu_P = 1$, and $s_k^0 = (2, 0)$, evading agents drive identical Dubins (or Reeds-Shepp) cars, $\mu_e = 0.5, \nu_e = 1$, $\xi = 100$, $\delta t = 10^{-3}$. The corresponding strategies are described by (18), (19), (27), (29). A pursuit is terminated at the states where the derivative of K^ξ is less than 10^{-3}. Besides, $\varphi_{E_1}^0 = 0$, $\varphi_{E_2}^0 = \pi/4$, $\mathcal{K}^\infty(s^0) = 2$, $\mathcal{K}^\xi(s^0) = 2.01391$, $\tau^\xi = 3.412$, $\mathcal{K}^\infty(s(\tau^\xi)) = 0.430907$, $K^\xi(s(\tau^\xi)) = 0.433905$ for Fig. 2, and $\varphi_{E_1}^0 = 0$, $\varphi_{E_2}^0 = \pi/4$, $\varphi_{E_3}^0 = -\pi/4$, $\mathcal{K}^\infty(s^0) = 2$, $\mathcal{K}^\xi(s^0) = 2.02209$, $\tau^\xi = 3.13$, $\mathcal{K}^\infty(s(\tau^\xi)) = 0.599302$, $\mathcal{K}^\xi(s(\tau^\xi)) = 0.60347$ for Fig. 3. These two examples demonstrate, in particular, that on the corresponding trajectories P definitely approaches the real target by $r = 0.5$ if only one decoy is launched, and P fails in the case of two decoys (see the capture areas bounded by dashed lines).

4. Conclusion

The paper describes a method for construction of locally optimizing strategies for games with terminal payoffs. It is assumed that the function whose value at the terminal state determines the payoff functional is defined everywhere in the game space and at least directionally differentiable. According to the common part of the method, the strategies are to meet the steepest descent/ascent conditions for this function. Some additional local conditions are invoked when the mentioned conditions do not allow to find all controls.

We apply the method to a class of games where the outcome equal to Euclidean distance to the furthest evading agent. A number of numerical experiments shows "expectable" behavior of the agents with plane kinematics described by some transition equations for wheeled robots (LaValle, 2006; Patsko and Turova, 2009). However, since for non-holonomic systems the approach involves additional assumptions, it is not clear if the designed strategies preserve guaranteed features.

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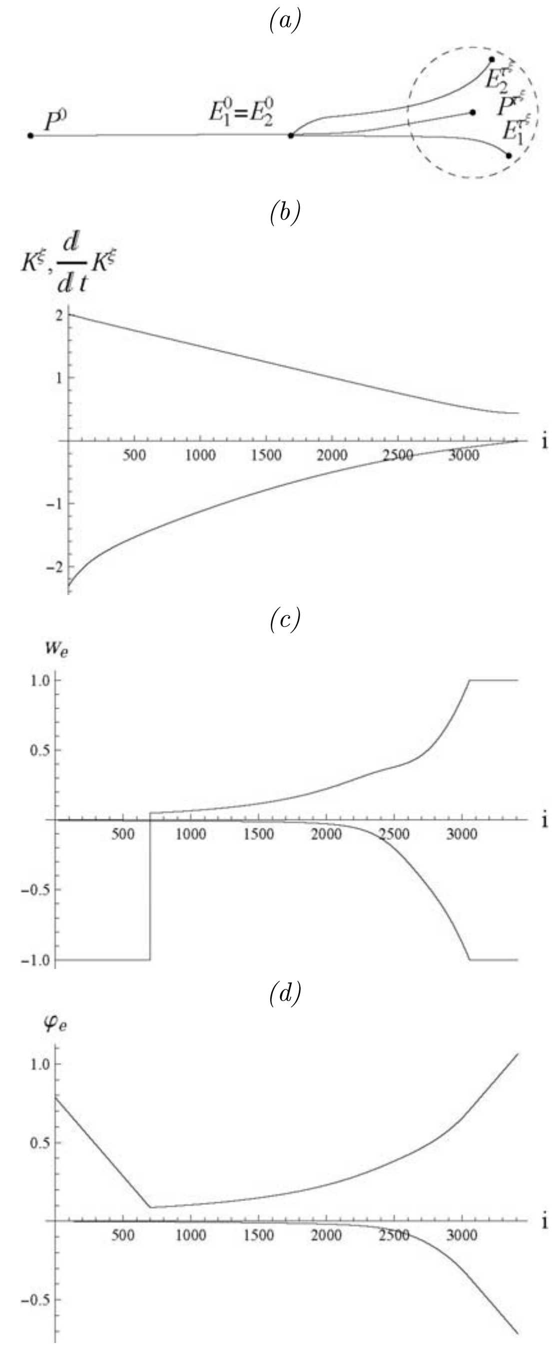

Figure2: Trajectories (a), evaluation function and its derivative (b), controls (c) and heading angles (d) as functions of i for one decoy

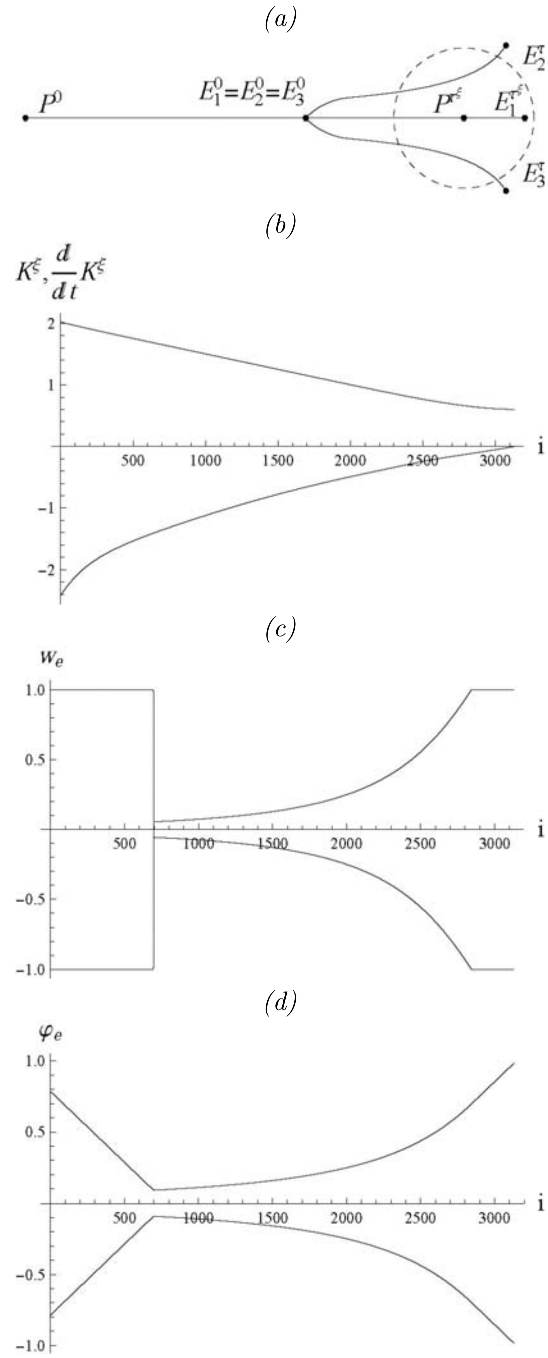

Figure3: Trajectories (a), evaluation function and its derivative (b), controls (c) and heading angles (d) as functions of i for two decoys

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Wisdom in Tian Ji's Horse Racing Strategy

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Abstract The philosophy behind Tian Ji's horse racing strategy, an interesting legendary Chinese story, has been analyzed mathematically. In this paper, the formulation of the generalized Tian Ji's horse racing strategy with different number of horses has been developed. Various winning, drawing or losing combinations and probabilities have been addressed. This may provide food for thought on business competitiveness or military struggle, in particular, the weaker overcoming the stronger.

Keywords: legendary Chinese story, decision making, Eulerian number.

1. Introduction

In the modern world today, the survival of the fittest holds for general people, but the survival of the cunning is possible for intellectuals. Nowadays, with the development of society, every company faces more intense competition, especially for weaker and smaller enterprises. In actuality, stronger and larger companies possess much more advantages and better talents than weaker and smaller ones. The situation applies to developing nations all over the world too. Every nation, every country, strong or weak, vies for a foothold in every arena of economy, sports, arts and so on. All aims to be the strongest and the best, but in actuality, the strong possesses much more advantages and better talents than the growing multitude of developing nations.

Despite the fact, the strongest of warriors has his Achilles' heel. In such a competitive world whereby almost everything can be rivaled in, it is necessary to plan the order of engagement or competition to overcome the stronger or to claim a greater victory. In economics, companies and government agencies allocate resources according to the needs and requirements of the situation, often giving the minimum budget for the success of a project to maximize profit. In another way, it is a way of conserving resources to cope with greater threats. On a personal level, when playing a group game, leader must choose his players in sequence. The sequence of the group game should be calculated properly, especially to encounter a stronger competitor. This procedure is very similar to Tian Ji's horse racing strategy, an interesting legendary Chinese story. In this paper, the generalized Tian Ji's horse racing strategy is analyzed mathematically with the aim of finding general clues which have the potential to be applied in the modern world today.

2. Tian Ji's Horse Racing Strategy

In ancient China, there was an era known as "Warring States Period" (403 BC – 221 BC) during which China was not a unified empire but divided by independent

Seven Warring States with conflicting interests, one of which was Qi State located in eastern China. From 356 BC to 320 BC, the ruler of Qi State was Tian Yin-Qi (378 BC – 320 BC), King Wei of Qi. The story of Tian Ji's horse racing strategy, which is well-known and popular in China today, was originally recorded (Si-Ma, 91 BC) in the biography of Sun Bin (? – 316 BC), as a military strategist in Qi State ruled by King Wei of Qi:

General Tian Ji, a high-ranked army commander in Qi State, frequently bet heavily on horse races with King Wei of Qi. Observing that their horses, divided into three different speed classes, were well-matched, Sun Bin then advised Tian Ji, "Go ahead and stake heavily! I shall see that you win." Taking Sun Bin at his word, Tian Ji bet a thousand gold pieces with the King. Just as the race was to start, Sun Bin counseled Tian Ji, "Pit your slow horse against the King's fast horse, your fast horse against the King's medium horse, and your medium horse against the King's slow horse." When all three horse races were finished, although Tian Ji lost the first race, his horses prevailed in the next two, in the end getting a thousand gold pieces from the King.

Amazedly, the victorious strategy (as did Tian Ji after following Sun Bin's advice) was remarkable to be solved 2300 years long before operations research and game theory were invented (Shu *et al.*, 2011). This was only one way that Tian Ji could claim a victory over the King, as illustrated in Fig. 1. All other ways would present Tian Ji with loss. Naturally, the Sun Bin's victorious advice, named Tian Ji's horse racing strategy, may be extended to the scenario that Tian Ji and the King would race horses with arbitrary N different speed classes. In order to facilitate the analysis of the generalized Tian Ji's N-horse racing strategy, the N horses owned by two players: Tian Ji (T) and the King (K) are denoted respectively by T_n and K_n, where the subscript $n = 1, 2, \cdots, N$ is defined as player T's or K's horse in the nth speed class. In this scenario of N-horse racing, T's horse in a faster class is able to beat K's horse in a slower class, but T's horse are unable to beat K's horse in the same or faster class. Without losing generality, the relative racing capabilities of horses are $T_{n+1} \prec K_{n+1} \prec T_n \prec K_n$ for any $n = 1, 2, \cdots, N-1$, where the symbol "$\prec$" means "unable to beat" and the larger subscript corresponds to the slower class.

Figure1: Tian Ji's horse racing strategy.

T and K would choose the same class of N-horse racing, that is, the pairwise racing is $\begin{pmatrix} T_1 & T_2 & \cdots & T_N \\ K_1 & K_2 & \cdots & K_N \end{pmatrix}$. Naturally, in view of T's horse slower than K's one in the same class ($T_n \prec K_n,\ n = 1, 2, \cdots, N$), T's horses would lose all races. The essence of Tian Ji's horse racing strategy is that the originally-classified racing appearance of T's horses should be shifted one place in order to achieve T's best result. The best strategy for the generalized Tian Ji's N-horse racing, suggested by Sun Bin, should be the pairwise racing $\begin{pmatrix} T_N & T_1 & \cdots & T_{N-1} \\ K_1 & K_2 & \cdots & K_N \end{pmatrix}$. T would claim a victory of the N-horse racing with one loss and $N-1$ wins.

If K always chooses the racing appearance $(K_1, K_2, \cdots, K_N)$ against the T's best response $(T_N, T_1, \cdots, T_{N-1})$, T would gain every bet. Naturally, K would soon realize that the racing appearance $(K_1, K_2, \cdots, K_N)$ is resulting in recurrent losses. K would become an active player and consider alternative racing appearance to turn the racing around. A competitive situation is encountered for each player competing with a total of $N!$ combinatorial pairwise racing $\begin{pmatrix} T_{\sigma(1)} & T_{\sigma(2)} & \cdots & T_{\sigma(N)} \\ K_1 & K_2 & \cdots & K_N \end{pmatrix}$ (where σ is the permutation) available to T and K. The above explanation indicates that T would lose all races for the unit permutation $\sigma(n) = n$ (as the T's worst strategy) and T would claim a victory with one loss and $N-1$ wins for the shift permutation $\sigma(n) = \begin{cases} N & n = 1 \\ n-1 & 1 < n \leq N \end{cases}$ (as the T's best strategy). Then the natural question is what is T's winning probability for randomly-pairwise racing between T's and K's horses. The equivalent question is how many permutations are available to T as T's victorious strategies.

Theorem 1. *The number of T having exactly M wins in N-horse racing is Eulerian number,*

$$E(N, M) = \sum_{m=0}^{M} (-1)^m (M+1-m)^N \frac{(N+1)!}{m!\,(N+1-m)!},$$

the number of permutations on $\{1, 2, \cdots, N\}$ with exactly M excedances.

Proof. An excedance of the permutation σ on $\{1, 2, \cdots, N\}$ is defined as any index n such that, $\sigma(n) > n$ and Eulerian number $E(N, M)$ (Euler, 1755) is defined as the number of the permutation σ on $\{1, 2, \cdots, N\}$ with exactly M excedances (Rosen, 2000). It is obvious that the existence condition of T having one win is $K_n \prec T_{\sigma(n)}$, that is, $\sigma(n) > n$ for any index n. To determine the number of T's wins is equivalent to count the number of $\sigma(n) > n$, which is an excedance of the permutation σ in the parlance of combinatorics. So the number of T having exactly M wins in N-horse racing is Eulerian number $E(N, M)$. The theorem is proved. □

In view of the symmetry property of Eulerian number, that is, $E(N, M) = E(N, N-(M+1))$, Theorem 1 can be expressed equivalently as

Theorem 2. *The number of T having exactly $M+1$ no-wins in N-horse racing is Eulerian number.*

In the above two theorems, it is interesting to note that the generalized Tian Ji's horse racing strategy, as the extension of the famous legendary Chinese story, can be viewed as a practical demonstration of applying Eulerian number. There are only three outcomes for T, namely, winning combination

$$\begin{cases} \sum_{M=\frac{N+1}{2}}^{N-1} E(N,M) & \text{odd } N \\ \frac{N!}{2} - E(N,\frac{N}{2}) & \text{even } N \end{cases}$$

with probability

$$\begin{cases} \frac{1}{N!}\sum_{M=\frac{N+1}{2}}^{N-1} E(N,M) & \text{odd } N \\ \frac{1}{2} - \frac{1}{N!}E(N,\frac{N}{2}) & \text{even } N \end{cases},$$

drawing combination

$$\begin{cases} 0 & \text{odd } N \\ E(N,\frac{N}{2}) & \text{even } N \end{cases}$$

with probability

$$\begin{cases} 0 & \text{odd } N \\ \frac{1}{N!}E(N,\frac{N}{2}) & \text{even } N \end{cases},$$

or losing combination

$$\begin{cases} \sum_{M=0}^{\frac{N-1}{2}} E(N,M) & \text{odd } N \\ \frac{N!}{2} & \text{even } N \end{cases}$$

with probability

$$\begin{cases} \frac{1}{N!}\sum_{M=0}^{\frac{N-1}{2}} E(N,M) & \text{odd } N \\ \frac{1}{2} & \text{even } N \end{cases},$$

which are shown in the table below.

Table1: Combination and probability with variable number of horses.

		Combination (with probability)		
N horses	Total $N!$	Winning	Drawing	Losing
1	1	0 (0%)	0 (0%)	1 (100%)
2	2	0 (0%)	1 (50%)	1 (50%)
3	6	1 (17%)	0 (0%)	5 (83%)
4	24	1 (4%)	11 (46%)	12 (50%)
5	120	27 (23%)	0 (0%)	93 (78%)
6	720	58 (8%)	302 (42%)	360 (50%)
7	5040	1312 (26%)	0 (0%)	3728 (74%)
8	40320	4541 (11%)	15619 (39%)	20160 (50%)
9	362880	103345 (29%)	0 (0%)	259535 (72%)
10	3628800	504046 (14%)	1310354 (36%)	1814400 (50%)

Probabilities for odd- or even- numbered horses are plotted respectively in Figs. 2 and 3. From the results illustrated above, the probabilities follow Eulerian distribution and there are two detectable characteristics. First, the case of odd-numbered horse racing has no drawing, which drawing happens only in the case of even-numbered horse racing. Second, the losing probabilities of any even-numbered horse racing are always at the constant 50% regardless of horse number involved.

Figure2: Trend of probability for odd-numbered horses.

In the odd-numbered horse racing, the winning and losing probabilities converge to the constant 50% as horse number increases due to no drawing; while in the even-numbered horse racing, the winning and drawing probabilities converge to a constant 25% as horse number increases due to the constant 50% losing. Overall, the winning probability of the odd-numbered horse racing is much higher than that of the adjacent even-numbered cases, for example, 23% of $N = 5$ is much higher than 4% of $N = 4$ and 8% of $N = 6$.

This shows that an odd-numbered horse racing gives better an opportunity of winning than an even-numbered case. This implies that the best combat units should be odd-numbered. Of course, if the combat efficiency would be getting better as N increases, the difficulty of controlling much large N combat units would be encountered. No drawing occurs in the odd-numbered horse racing, which means that a decisive outcome must be reached instead of a stalemate. More importantly, Figs. 2 and 3 suggest that the more horses involved, the larger N, result in the higher winning probability. Philosophically, it is typically the epitome of winning in numbers.

3. Conclusion

This paper discusses the formulation of determining the winning, drawing and losing probabilities of the generalized Tian Ji's horse racing strategy for any given racing horse numbers. Based on Eulerian number, the way of calculating the combination of having M wins in N-horse racing is so straightforward, thereby enabling us to find the probability of winning an entire game by having more wins than losses. It

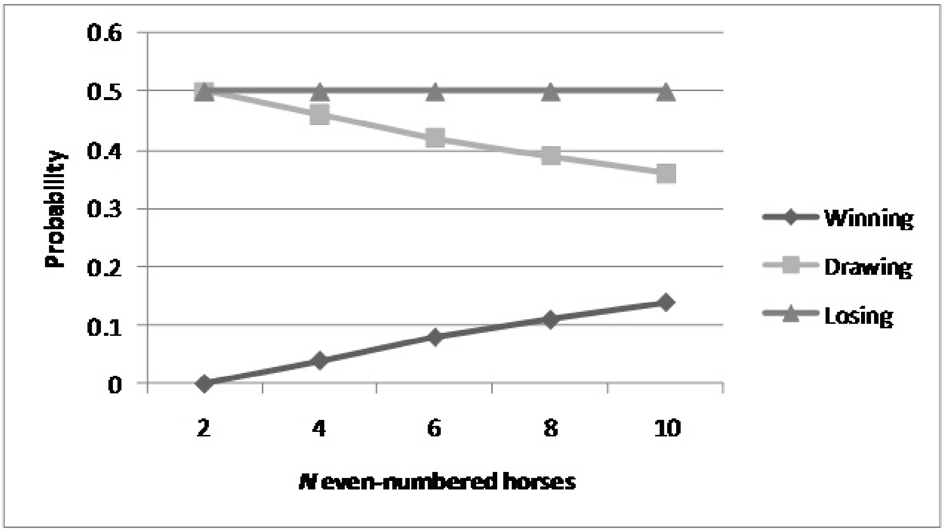

Figure3: Trend of probability for even-numbered horses.

is worth to mention to this end that that the larger N results in the higher winning probability. Philosophically, it is typically the epitome of winning in numbers.

Tian Ji's horse racing strategy is an interesting legendary Chinese story, which gives valuable insights to intellectuals that nothing is absolutely certain and thus nothing is impossible. Studying the theory of Tian Ji's horse racing strategy is very beneficial to society. In sports, this could be used in matching competitors in group games, such as, tennis, football, table tennis, badminton, *etc.* In economics, it could be used to allocate minimum resources to suitable tasks for optimizing profits. In engineering logistics, the results could be applied to arrange and transport goods with limited vehicles or manpower.

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Differential Game of Pollution Control with Overlapping Generations

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Abstract We formulate an overlapping generations model on optimal emissions with continuous age-structure. We compare the non-cooperative solution to the cooperative one and obtain fundamental differences in the optimal strategies. Also including an altruistic motive does not avoid the problem of the *myopic* non-cooperative solution. Finally we define a time-consistent tax scheme to obtain the cooperative solution in the non-cooperative case.

Keywords: differential game, overlapping generations, pollution, Pontryagin's Maximum Principle, Nash equilibrium

1. Introduction

Environmental topics are a classical topic where game theory and differential games are used to study the strategic interaction between agents. Examples are Breton et al. (2005) where agents maximize their utility by producing revenue which causes emissions. The model uses a finite number of symmetric agents who play the game over a finite deterministic time horizon T. The game is generalized in Shevkoplyas and Kostyunin (2011) where the terminal time T is stochastic.

The above mentioned papers as well as many other papers use players that are symmetric in their age and in their duration of the game. However, this is a significant restriction to the model because of the following reasons. i) If a model uses natural persons as players it is a considerable restriction to ignore the survival schedule of the persons. Individuals are born, age, die and their life-horizon is finite. Further, the set of players who are acting is age-structured. So players act against players in all other age groups. Moreover the preferences of the players possibly change over their age. Thus it is natural to assume that the strategies of the players will change according to their age. ii) If a model uses representative players with infinite time horizon the players will include the whole time horizon (no finite restriction to time) in their optimization. If a model on the other hand uses representative players with finite time horizon (the life-cycle is finite) t he players will pollute the environment more when they are nearing their end of life. However, in reality the story lies in between. The agents should have a finite life, but the environment exists forever.

To overcome the above two points we use an overlapping generations model with continuous age-structure. That allows to include the finite life-time of the agents who act over their own life-cycle, but pollute the environment which evolves over time forever. With this simple model we can address a very important topic. Many players act over a finite life-time and pollute the environment. Therefore later living cohorts will suffer from that *myopic* behaviour of earlier cohorts. Further we can illustrate and provide an interpretation of the difference between the optimal strategies in the non-cooperative and the cooperative solution. Following Barro and Becker (1989) we also include the possiblity of altruism. All players are not only interested in their own utility but also in the utility of their surviving spouses. The result shows that this motive decreases optimal emissions, but cannot turn the result into the cooperative one.

Up to the best of our knowledge the first differential game model using an overlapping generation structure is Jorgensen and Yeung (1999) in a renewable resource extraction context. The model uses an overlapping generation framework where new cohorts enter the model at discrete points in time. The players exploit a stock of renewable resources. This research has been followed by Grilli (2008). For a cooperative solution and a stochastic extension of the model we refer to Jorgensen and Yeung (2001) and Jorgensen and Yeung (2005). Hierarchical structure (i.e. Stackelberg) to this setup has been introduced in Grilli (2009).

The rest of the paper is organized as follows. Section 2. presents the formal definition of the model as well as corresponding assumptions. The model is solved in section 3. according to the open-loop Nash framework. In contrast to that in section 4. we derive the cooperative solution and provide a comparison to the previous section. To overcome inefficiencies we introduce a tax scheme in section 5.. Section 6. concludes.

2. The Model

At each point in time t one generation is born (i.e. enters the model). The size of the generation is $n(0,t)$ which is assumed to be exogenous and constant over time, i.e. $n(0,t) = n$ for $\forall\, t$. The individuals are assumed to be symmetric and live and die according to an exogenously given survival probability $S(a)$ which is equal for all cohorts.[1] We further assume that the maximal life-span is ω for all individuals of all cohorts. Note that this is not a restriction to the model, as we can choose ω arbitrarily large (e.g. 300 years), such that all individuals have died (exit the game) after ω. Thus we have $S(\omega) = 0$ (for details on a formal condition we refer to Anita (2001)) and trivially $S(0) = 1$.

The game is formulated over a finite time horizon $T > \omega$, which can be choosen arbitrarily high. As a result the game has to be considered over time $t \in [0,T]$ and age $a \in [0,\omega]$. The set of players participating in the game is age-structured. At each point in time $\int_0^\omega n(a,t)\ da = n \int_0^\omega S(a)\ da$ players participate in the game, where $n(a,t)$ denotes the number of a-year old individuals at time t. Note that this number is a constant[2] since we assume an exogenous and constant survival

[1] It is straightforward to introduce an exogenous time dependent survival probability into the model. The expressions will not change. Moreover it would also be interesting to introduce an endogenous survival probability that depends on the quality of the environment (i.e. on the stock of pollution). Depending on the form of the endogenous interaction this is possibly quite involved.

[2] To be mathematically precise: It is constant (i) in general after time $t = \omega$ or (ii) in particular from the beginning when the initial distribution of players is choosen according to the survival probability, i.e. $n(a,0) = nS(a)$.

probability, which is independent of time, and an exogenous and constant number of newborns. The game is started at $t = 0$ with the initial age-distribution of players $n(a, 0)$.

Remark on the notation: We use $(a, t) \in [0, \omega] \times [0, T]$ to denote variables or parameters that depend on age a and time t. However, many calculations and expressions are made over the life-cycle of one player. In that case we use the notation $(a, t_0 + a)$ (age and time advance with the same pace), t_0 denotes the time of birth of that player.

Each player derives utility from own emissions $e(a, t)$. These individual emissions aggregate and pollute the environment. On the other hand the environment regenerates with an exogenous given rate $\delta \geq 0$. Thus the stock of pollution $P(t)$ changes over time according to the following dynamics

$$\dot{P}(t) = \int_0^\omega S(a)e(a,t) \; da - \delta P(t), P(0) = P_0 \tag{1}$$

For simplicity in later calculations we define the total emissions at time t as

$$E(t) := \int_0^\omega S(a)e(a,t) \; da \tag{2}$$

implying $\dot{P}(t) = E(t) - \delta P(t)$.

As already mentioned the players derive utility from own emissions according to the strictly concave instantaneous utility function $R(e(a,t))$. On the other hand their well being is diminished by a negative effect of the pollution stock $d(a,t)P(t)$, where $d(a,t)$ might depend on age (e.g. young or old individuals suffer more hard from pollution) and time (e.g. technological development in protection against pollution). Finally the players have an altruistic motive. They face positive utility from the utility of surviving spouses. This is denoted by $Q(a,t)$, which is defined as

$$Q(a,t) := \int_0^a \nu(a-s, t-s)S(s)\big(R(e(s,t)) - d(s,t)P(t)\big) \; ds \tag{3}$$

where $\nu(\cdot)$ denotes their age-structured (exogenous) fertility rate.[3] $\nu(a-s, t-s)S(s)$ denotes the number of s-year old spouses at time t and $\big(R(e(s,t)) - d(s,t)P(t)\big)$ denotes their corresponding utility at current time t (note that the alturistic utility of the spouses is not included here). Thus, $Q(a,t)$ aggregates the utility of the surviving spouses (of one player).

Each player maximizes his own utility over his life-cycle, where each period is weighted by the survival probability. Thus the objective function of a player born at t_0 faces the following objective function[4]

$$\max_{e(\cdot)} \int_0^\omega e^{-ra} S(a)\Big(R(e(a,t_0+a)) - d(a,t_0+a)P(t_0+a) + \gamma Q(a,t_0+a)\Big) \; da \tag{4}$$

[3] Since we assume a survival schedule $S(a)$ for $a \in [0, \omega]$ that is constant over time and a constant number of newborns n, also the age-structured fertility rate has to be constant over time, i.e. $\nu(a,t) = \nu(a)$ for $\forall \; [0, \omega]$. Further the Lotka equation has to hold, i.e. $\int_0^\omega \nu(a)S(a) \; da = 1$ (where Lotka's r is zero in case of a stationary population).

[4] To be mathematically precise: (4) denotes the objective function of a player whose entire life-span lies within $[0, T]$, i.e. $t_0 \geq 0$ and $t_0 + \omega \leq T$. For the other players the objective function has to be adapted slightly.

r is the time preference rate, which is assumed to be equal for all players. $\gamma \in [0,1]$ is a parameter that determines the level of the altruism. Thus $\gamma = 0$ implies a model without altruism.

3. Non-Cooperative Solution

Within this section we begin with the derivation of the open-loop Nash equilibrium and show that it is subgame perfect and that it coincides with the feedback Stackelberg solution. In order to calculate the open-loop Nash equilibrium we have to use Pontryagin's Maximum Principle (PMP) (see e.g. Grass et al., 2008 or Feichtinger et al., 1986). The derivations are presented for a player born at time t_0 representatively.

We formulate the current value Hamiltonian for a player born at t_0 (in the following we skip the age and time indexes if this does not lead to confusion)

$$\mathcal{H}^{ON} = S(R(e) - dP + \gamma Q) + \lambda^P(E - \delta P) \tag{5}$$

where λ denotes the adjoint variable of the stock of pollution.

The first order condition (for an inner solution) with respect to the control variable reads

$$\begin{aligned}\mathcal{H}_e^{ON} &= SR'(e)(1 + \nu(0, t-a)S) + \lambda^P S \\ &= SR'(e) + \lambda^P S = 0\end{aligned} \tag{6}$$

where the second equality follows from the fact that we assume that $\nu(0, t-a) = 0$, i.e. there is no fertility at age 0. This assumption is common in demographics.

For the dynamics of the adjoint equation we obtain[5]

$$\dot{\lambda}^P = (r+\delta)\lambda^P + S\Big(d + \gamma \int_0^a \nu(a-s, t-s)S(s,t)d(s,t)\ ds\Big) \tag{7}$$

This can be solved by using the transversality condition $\lambda^P(\omega, t) = 0$ (no salvage value)

$$\begin{aligned}\lambda^P(a,t) = -\int_a^\omega e^{-(r+\delta)(s-a)} S(s, t-a+s)\Big[d(s, t-a+s) \\ +\gamma \int_0^s \nu(s-s', t-a+s-s')S(s', t-a+s)d(s', t-a+s)\ ds'\Big]\ ds\end{aligned} \tag{8}$$

From the abve expression it is clear that $\lambda^P(a,t) < 0$ for $\forall\ a \in [0, \omega)$. Intuitively this is straightforward, as the stock of pollution diminishes the utility of the players. There is no positive effect of $P(t)$.

Applying the revenue function of Shevkoplyas and Kostyunin (2011), i.e. $R(e) = e(b - \frac{e}{2})$, for the first order condition (6) we obtain the following optimal emissions

$$e(a,t) = \begin{cases} b - \int_a^\omega e^{-(r+\delta)(s-a)} S\Big[d + \gamma \int_0^s \nu S d\ ds'\Big]\ ds & \text{if } -\lambda^P(a,t) < b \\ 0 & \text{else} \end{cases} \tag{9}$$

[5] Within this section we use the dot-notation for $\dot{\lambda}^P(a,t) = \frac{d\lambda(a, t_0+a)}{da}$. The dot-describes the derrivative along the life-cycle of a player.

Thus emissions are positive as long as the marginal damage of total emissions is smaller than the linear part of the instantaneous utility function. This is different when we apply $R(e) = 2\sqrt{e}$ (here the Inada condition is fulfilled $\lim_{e \to 0+} = +\infty$),

$$e(a,t) = \big(-\lambda^P(a,t)\big)^{-2}. \tag{10}$$

Here optimal emissions are always positive and increasing over the life-cycle of an individual (note that the adjoint variable decreases over the life-cycle).

Having derived the optimal emissions explicitely (for different instantaneous utility functions) we can think about the intuition and interpretation of the outcome:

i) Both expressions show that optimal emissions only depend on exogenous parameters. They do not depend on the emissions of other players, on time and on the state (also not on the initial condition). Thus it is obvious that the solution is subgame perfect. Moreover every cohort behaves in the same way as long as we assume that the cohorts are equal with respect to the survival probability, their fertility rate and thier preferences. When we would assume a change in the population structure also the optimal strategies of different cohorts would be different.

ii) We see for both expressions that the altruistic motive diminishes optimal emissions. The higher the level of altruism is (the higher γ) the lower the emissions will be. However, this does not answer the question whether including altruism lead to a socially optimal outcome (or whether altruism overcompensates the gap). We will come back to that topic in the next section, where the cooperative solution is derived.

iii) For the time derrivative of the emissions we obtain

$$\begin{aligned}\dot{e}(a,t_0+a) &= \frac{-\dot{\lambda}^P(a,t_0+a)}{R''(e)} \\ &= -\frac{1}{R''(e)}\Big((r+\delta)\lambda^P + S\big(d + \gamma\int_a^\omega vSd\ \ ds\big)\Big)\end{aligned} \tag{11}$$

The sign of the above expression is unambiguous. However, for sufficiently small values of r and δ it is positive, as we assume a concave revenue function $R(e)$. Therefore for that case the emissions are strictly increasing over the life-cycle of an individual. However also for general values of r and δ the above expression will be positive for most age groups, since $\lambda^P(a,t_0+a) < 0$ for $a < \omega$ and $\lambda^P(\omega,t_0+\omega) = 0$. This reflects that life is finite. The nearer an individual gets to its death, the less s/he is interested in the pollution stock, as her/his time to suffer from it gets shorter. This is something that stands in contrast to models where the population has no age-structure. The derrivatives of optimal emissions with respect to age and time only are trivial, i.e.

$$\begin{aligned}\frac{de(a,t)}{da} &= \frac{-\dot{\lambda}^P(a,t_0+a)}{R''(e)} \\ \frac{de(a,t)}{dt} &= 0\end{aligned} \tag{12}$$

In this section we have calculated the open-loop Nash equilibrium of the game. Since the optimal strategies only depend on exogenous terms (not on the strategies of the opponents, not on the state, and not on the initial conditions), the resulting equilibrium is also subgame perfect. Nevertheless, so far we have not dealt with the question what happens if the differential game has hierarchical structure (i.e. Stackelberg). Because of the age-structure in the players it can be justified that e.g. the *old* players (i.e. the players that are older than a certain treshhold age $0 < \bar{a} < \omega$) at each time instant take up the leadership. The younger players then would act as followers. For an example on such a hierarchical structure we refer to the resource extraction model by Grilli (2009). If such a hierarchical structure is assumed in our model (or any other hierarchical structure, e.g. some generations act as leaders, etc.) it is not necessary to derive the feedback Stackelberg equilibrium. This follows from the fact that our differential game is additively separable in all controls and state-redundant. Bacchiega et al. (2008) show that this implies that the open-loop and feedback Nash equilibrium coincide and that further the feedback Nash equilibrium coincides with the feedback Stackelberg equilibrium. Putting into other terms this means that there is no difference whether we assume a hierarchical structure to our differential game. The (open-loop Nash and the feedback Stackelberg) results will not change.

4. Cooperative Solution

Having dealth with non-cooperative solutions in the previous section we turn to the cooperative one in this section. As our differential game uses an overlapping generations framework we have to use the age-structured PMP presented in Feichtinger et al. (2003) or Brokate (1985).

The solution of the cooperative differential game is the solution of the following problem,

$$
\begin{aligned}
\max_{e(a,t)} \quad & \int_0^T \int_0^\omega e^{-rt} S(a)\Big[R(e(a,t)) - d(a,t)P(a,t) + \gamma Q(a,t)\Big] \; da \; dt \\
P_a + P_t = & \int_0^\omega S(a)e(a,t) \; da - \delta P(t) \\
& P(0,t) = \bar{P}(t) = \frac{1}{\omega}\int_0^\omega P(a,t) \; da, P(a,0) = P_0 \\
E(t) = & \int_0^\omega S(a)e(a,t) \; da \\
Q(a,t) = & \int_0^a \nu(a-s,t-s)S(s)\Big[R(e(s,t)) - d(s,t)P(t)\Big] \; ds
\end{aligned}
\tag{13}
$$

Remark: Although the stock of pollution does only depend on time, it is formulated as age and time dependent in the above formulation. This transformation is artificial since it is equal for each age group, i.e. $P(a,t) = P(s,t)$ for $\forall\ a, s \in [0,\omega]$. The additional condition $P(0,t) = \bar{P}(t)$ is necessary such that $P(0,t) = P(a,t)$ for $\forall\ a \in [0,\omega]$ holds. This formulation allows to apply the standard form of the age-structured PMP (see Feichtinger et al. (2003)). Moreover it allows to calculate the adjoint variable in the analogous way to that in the previous section, which makes them easier to compare.

The current value Hamiltonian for this problem reads (we again skip the age and time argument whenever they are not of particular importance)

$$\mathcal{H}^C = S(R(e) - dP + \gamma Q) + \xi^P(E - \delta P) + \eta^E Se + \eta^{\bar{P}} \frac{1}{\omega} P \tag{14}$$

For the first order condition (for an inner solution) we obtain

$$\begin{aligned} \mathcal{H}^C_e &= SR'(e)(1 + \nu(0, t - a)) + \eta^E S \\ &= SR'(e) + \eta^E S = 0 \end{aligned} \tag{15}$$

where the second equality again follows from the fact that we assume zero fertility rate at birth.

For the dynamics of the adjoint equations we obtain

$$\begin{aligned} \xi^P_a + \xi^P_t &= (r + \delta)\xi^P + S\Big(d + \gamma \int_0^a \nu(a - s, t - s) dS(s, t)\ ds\Big) - \frac{1}{\omega}\eta^{\bar{P}} \\ \eta^E &= \int_0^\omega \xi^P\ da \\ \eta^{\bar{P}} &= \xi^P(0, t) \end{aligned} \tag{16}$$

Using the transversality condition $\xi^P(\omega, t) = 0$ we obtain

$$\begin{aligned} \xi^P(a, t) = &- \int_a^\omega e^{-(r+\delta)(s-a)} \Big[S(s, t - a + s) \times \\ &\Big(d + \gamma \int_0^s \nu(s - s', t - a + s - s') dS(s', t - a + s)\Big)\ ds' - \\ &\frac{1}{\omega}\xi^P(0, t - a + s)\Big]\ ds \\ = &\ \lambda^P(a, t) + \frac{1}{\omega} \int_a^\omega e^{-(r+\delta)(s-a)} \xi^P(0, t - a + s)\ ds \end{aligned} \tag{17}$$

for $t - a + \omega \le T$. For the case $t - a + \omega > T$ the expression is analogous, only the bounds of the integrals have to be choosen correspondingly.

Together with the transversality conditions $\xi^P(\omega, t) = 0$ and $\xi^P(a, T) = 0$ we can show that $\xi^P(a, t) < \lambda^P(a, t) < 0$ holds for $\forall\ a \in [0, \omega)$. I.e. the marignal damage of the stock of pollution for an a-year old player at time t is stronger in case of the cooperative solution. This reflects that the objective function also values cohorts that are born later on.

Again we derive the optimal emissions for two choices of the revenue function. By assuming $R(e) = e(b - \frac{e}{2})$ we obtain

$$e(a, t) = \begin{cases} b + \int_0^\omega \Big(\lambda^P(a, t) + \frac{1}{\omega} \int_a^\omega e^{-(r+\delta)(s-a)} \xi^P(0, t - a + s)\ ds\Big)\ da, \\ \quad \text{if } -\int_0^\omega \xi^P(a, t)\ da < b \\ 0, \quad \text{else} \end{cases} \tag{18}$$

By applying $R(e) = 2\sqrt{e}$ (the Inada condition is fulfilled $\lim_{e \to 0+} = +\infty$) we have

$$\begin{aligned} e(a, t) &= \Big(- \int_0^\omega \xi^P(a, t)\ da\Big)^{-2} \\ &= \Big[- \int_0^\omega \Big(\lambda^P(a, t) + \frac{1}{\omega} \int_a^\omega e^{-(r+\delta)(s-a)} \xi^P(0, t - a + s)\ ds\Big)\ da\Big]^{-2} \end{aligned} \tag{19}$$

which is always positive. In both cases the optimal emissions does not depend on age but on time. Formally this can be written as

$$
\begin{aligned}
&e_a(a,t) = 0 \\
&e_t(a,t) = -\frac{1}{R''(e)} \int_0^{\omega} (r+\delta)\xi^P + S\Big(d + \gamma \int_0^a \nu dS \;\; ds\Big) \;\; da
\end{aligned} \tag{20}
$$

At every time instant optimal emissions are equal for all participating players (independently of their age). But the emissions increase over time. A comparison to the non-cooperative solution shows a fundamental difference, summarized in Table 1.

Table1: Comparison: non-cooperative vs. cooperative solution.

non-cooperative	cooperative
$e_a(a,t) > 0$	$e_a(a,t) = 0$
$e_t(a,t) = 0$	$e_t(a,t) > 0$

The intuition is the following. In both cases emissions increase, but over a different horizon.

- In the non-cooperative solution the emissions increase over the life-cycle, where every player behave the same. I.e. it does not matter when a player is born, at age a s/he will have the same optimal emissions. This is a result of the *myopic* optimization of the own life and of the linear damage of the stock of pollution. The stock of pollution at birth is irrelevant and additively seperable from the 'new' pollution over the life-cycle. In case of a non-linear damage of the stock of pollution $d(a,t,P(t))$ there will be differences between generations (compared at the same age). The altruistic motive diminishes the emissions, but not enough to turn them into a shape that is socially optimal.
- In the cooperative solution the story is the other way around. The utility of all players is optimized once. So it is not optimal that players behave completely selfish until the end of their life ('Behind me there is nothing!'). The emissions do also increase over time, but for all ages simultaneously. This solution is therefore a tradeoff such that players living early in the time horizon have relatively low emissions, but a high quality environment. And the later the time gets the higher the emissions will be, to compensate for the lower quality of the environment. In this case the results will be qualitatively the same in case of a non-linear damage function (of the stock of pollution).

The comparison of the two solutions is something that is not possible without assuming the age-structure in the set of players. However, it is important to learn how pollution works in the real world. And that including altruism (what is usually not considered) is not enough to reach a solution that is socially optimal.

In the following section we address the question what can be done to turn the emissions in a non-cooperative solution to the socially optimal ones.

5. Time-consistent tax scheme

In order to obtain the cooperative optimal solution in the non-cooperative game we introduce taxes on emissions in order to provide an incentive to behave optimally from a cooperative point of view. The idea is that the players have to pay a tax rate τ for their emissions. Thus their objective function (4) turns into

$$\max_{e(\cdot)} \int_0^\omega e^{-ra} S(a)\Big(R(e(a,t_0+a)) - \tau e(a,t_0+a) - d(a,t_0+a)P(t_0+a) + \gamma Q(a,t_0+a)\Big) da \tag{21}$$

Since the states and the controls are separable the dynamics of the state and the adjoint variables does not change. However, the first order condition for the emissions has to be adapted. We obtain

$$\mathcal{H}_e^{ON\tau} = SR'(e) - Se\tau + \lambda^{P\tau} S \tag{22}$$

Using then the first order condition of the cooperative solution (15) we then obtain the tax rate

$$\tau(a,t) = \lambda^P(a,t) - \int_0^\omega \Big(\lambda^P(a,t) + \frac{1}{\omega}\int_a^\omega e^{-(r+\delta)(s-a)}\xi^P(0,t-a+s)\ ds\Big)\ da \tag{23}$$

which is age-structured now, as the the optimal emissions in the non-cooperative solution depend on the age of the player, whereas they are constant in the cooperative solution. With the same arguments we used in section 3. we can argue that the resulting solution is again subgame perfect.

Note that this result is important from a political point of view. Nowadays emissions and the resulting climate change is a big issue, but there are many different opinions around how to reduce emissions on a total level. In our small model we can argue that the damage of the stock of pollution refers to the damage that is caused by the climate change. The message is that the emissions will be much higher when every player maximizes only over the own life compared to a cooperative solution. Even including altuism into the model cannot solve the problems. However, it is possible to introduce a tax on emissions that turns the non-cooperative result into the cooperative one (in a subgame perfect way).

6. Conclusions

We have formulated an overlapping generations differential game on optimal emissions with continuous age-structure. We derived the non-cooperative solution in the open-loop Nash form and show that it is subgame perfect and equals the feedback Stackelberg solution. By the comparison to the cooperative solution we can derive the differences in the optimal strategies of the two solution forms. Including an altruistic motive does not turn the non-cooperative solution into the cooperative one.

The results of the model show that including age-structure implies different results to models with representative agents over time. This illustrates that age-structure is very important in many contexts and provides more realistic results, whenever age or the finite life-time is a key feature in the model (e.g. resource extraction, taxes, health). As a result it can be expected that further development of age-structured differential games and application to other model will provide very interesting new results.

The model can be extended in a couple of directions. First, the damage of the stock of pollution should be allowed to be a general (non-linear) function. Convex, concave as well as other forms are possible. Second, the cooperative solution should be extended to the infinite time horizon. In this case it is interesting to derive the condition and level for a steady state. Further the difference to the above case with the finite time horizon is interesting and will propose important conclusions from a political point of view (politicians act usually up to a finite time horizon). Third, it is realistic to assume that the stock of pollution influences the health of the players. I.e. the survival probability as well as the fertility rate should depend on $P(t)$. This is interesting because of the interpretation, as well as from a game theoretic point of view, since the strategies of the players influence the number of players in the future (during and after the own life). Finally, it is very interesting to inctroduce a second type of player (i.e. the government) who fixes the taxes. This player then has a different time horizon than the other players (and a different objective function). In the current model we have derived the taxes such that the non-cooperative result equals that of the cooperative sulution. In the case where the government fixes the taxes, it will also depend on their objective function. Also this extension is interesting from a methodological point of view.

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Consistent Subsolutions of the Least Core*

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Abstract The least core, a well-known solution concept in TU games setting, satisfies many properties used in axiomatizations of TU game solutions: it is efficient, anonymous, covariant, possesses shift-invariance, and max-invariance. However, it is not consistent thought the prenucleolus, that is consistent, is contained in it. Therefore, the least core may contain other consistent subsolutions. Since the union of consistent in the sense of Davis–Maschler solutions is also consistent, there should exist the unique *maximal under inclusion* consistent subsolution of the least core. In the paper we present and characterize this solution.

Keywords: Cooperative game, least core, prekernel, consistency

1. Introduction

Consistency properties of solutions for game with transferable utilities (TU games) connect between themselves the solution sets of games with different player sets. This property means that given a TU game and a coalition of players leaving the game with payoffs prescribed them by a solution, the other players involved in a *reduced game*, should obtain, in accordance with the same solution, the same payoffs as in the initial game.

This property is a powerful tool in the study of social welfare functions and orderings. However, in cooperative game setting the reduced games are not defined uniquely by both TU game and solution concept. There are some approaches to the definition of the reduced games and the corresponding to them definition of consistency, The first and the most popular definition belongs to Davis and Maschler (Davis and Maschler, 1965), who defined the characteristic function of the reduced game in the assumption that the players leaving the game gave up all their power to the remaining coalitions. Just this definition of the reduced games will be used in the paper.

There are some other properties of TU game solutions which the most well-known solutions possess: they are efficiency, anonymity (equal treatment property (ETP)), and covariance under strategic transformations. Together with consistency the unique maximal under inclusion solution satisfying them is the *prekernel* (Davis and Maschler, 1965), and the unique single-valued solution, i.e., a minimal one under inclusion, is the *prenucleolus*. It is worthwhile to study other non-empty, efficient, anonymous (or/and satisfying ETP), covariant, and consistent solutions for the class of all TU games. It is clear that all of them are contained in the prekernel.

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We begin to study such solutions from those that are contained in the *least core* as well. The least core is a well-known solution concept for TU games. The least core in non-empty for every TU game, it is efficient, anonymous, and covariant. Moreover, it is contained in the core when the latter is non-empty. However, the least core is not consistent. Nevertheless, it deserves studying, since it turns out to be the first step for finding the prenucleolus, as the result of minimization of the maximal excesses of coalitions. Perhaps, there are many consistent subsolutions of the least core. Since the union of consistent solutions is also consistent, it is natural, first, to describe the unique maximal under inclusion consistent subsolution of the least core. In section 2 the necessary definitions and the known properties of some solution are given. Section 3 gives a recurrent in the number of players formula for the membership of a payoff vector to a consistent subsolution of the least core. The main results are contained in Sections 4 and 5, where a combinatorial and an axiomatic characterizations of the maximal consistent subsolution of the least core are given respectively. Examples are collected in section 6.

2. Definitions and known results

Let $\mathcal{N}$ be a set (the universe of players), then a *cooperative game with transferable utilities (TU game)* is a pair (N, v), where $N \subset \mathcal{N}$ is a finite set, the set of players, and $v : 2^N \to \mathbb{R}^1$ is a *characteristic function* assigning to each coalition $S \subset N$ a real number $v(S)$ (with a convention $v(\emptyset) = 0$), reflecting a power of the coalition. In the sequel we consider the class of all TU games $\mathcal{G}_{\mathcal{N}}$ for some universe $\mathcal{N}$.

For any $x \in \mathbb{R}^N$, $S \subset N$ we denote by x_S the projection of x on the space $\mathbb{R}^S$, and by $x(S)$ the sum $\sum_{i\in S} x_i$, with a convention $x(\emptyset) = 0$.

A *solution* σ is a mapping associating with each game (N, v) a subset $\sigma(N, v) \subset X(N, v)$ of its *feasible payoff vectors*

$$X(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i \le v(N)\}.$$

By $X^*(N, v)$ we denote the set of *efficient* payoff vectors or *preimputations*:

$$X^*(N, v) = \{x \in \mathbb{R}^N \mid \sum_{i\in N} x_i = v(N)\}.$$

If for each game (N, v) $|\sigma(N, v)| = 1$, then the solution σ is called *single-valued (SV)*.

If for a game (N, v) the equalities $v(S \cup \{i\}) = v(S \cup \{j\})$ hold for some $i, j \in N$ and all $S \subset N \setminus \{i, j\}$, then the players i, j are called *substitutes*.

Recall some well-known properties of TU games solutions:

A solution σ

- is *non-empty* or satisfies *nonemptiness (NE)*, if $\sigma(N, v) \neq \emptyset$ for every game (N, v);
- *efficient,* or *Pareto optimal (PO)*, if $\sum_{i\in N} x_i = v(N)$ for every $x \in \sigma(N, v)$ and for every game (N, v);
- is *anonymous (ANO),* Let $(N, v), (M, w)$ be arbitrary games. If there exists a bijection $\pi : N \to M$ such that $\pi v = w$, where $\pi v(S) = v(\pi^{-1}(S)) \,\forall S \subset N$, then $\sigma(M, w) = \pi\sigma(N, v)$, (where, for any $x \in \mathbb{R}R^N$, $(\pi x)_j = x_{\pi^{-1}(j)} \,\forall j \in M$);

- satisfies the *equal treatment property (ETP)*, if for every game (N, v), for every $x \in \sigma(N, v)$ it holds $x_i = x_j$ for all substitutes $i, j \in N$;
- is *covariant (COV)*, if it is covariant under strategical transformations of games:

$$\sigma(N, \alpha v + \beta) = \alpha\sigma(N, v) + \beta$$

  for all $\alpha > 0, \beta \in \mathbb{R}^N$, where $(\alpha v + \beta)(S) = \alpha v(S) + \sum_{i \in S} \beta_i$ for all $S \in N$;
- is *shift invariant (SHI)*, if for every game (N, v) and number α $\sigma(N, v) = \sigma(N, v + \alpha)$, where

$$(v + \alpha)(S) = \begin{cases} v(N), & \text{if } S = N, \\ v(S) + \alpha & \text{for } S \subseteqq N. \end{cases};$$

- is *individual rational*, if from $x \in \sigma(N, v)$ it follows $x_i \geq v)\{i\})$ for all $k \in N$;
- is *consistent (CONS)* (Sobolev, 1975,Peleg, 1986), if for any game (N, v) and $x \in \sigma(N, v)$ it holds that

$$x_{N \setminus T} \in \sigma(N \setminus T, v^x_{N \setminus T}), \tag{1}$$

  where $(N \setminus T, v^x_{N \setminus T})$ is the *reduced game* being obtained when a coalition $T \subsetneqq N$ leaves the game;
- is *bilateral consistent*, if the previous property takes place for two-person reduced games ($|N \setminus T| = 2$;)
- is *converse consistent (CCONS)* (Peleg, 1986), if for every game (N, v) with $|N| \geq 2$, and $x \in X^*(N, v)$
  from $x_{\{i,j\}} \in \sigma(\{i, j\}, v^x_{\{i,j\}})$ for all $i, j \in N$ it follows $x \in \sigma(N, v)$;

The last four properties need the definition of reduced games. Different definitions of the reduced games lead to different definitions of consistency. In the paper we will rely the so-called "max" consistency which defines the reduced games in the sense of Davis–Maschler (Davis and Maschler, 1965).

Given a TU game (N, v), its payoff vector $x \in X(N, v)$, and a coalition $S \subsetneqq N$, the Davis–Maschler *reduced game* (Davis and Maschler, 1965) (S, v^x_S) on the player set S with respect to x is defined by the following characteristic function

$$v^x_S(T) = \begin{cases} v(N) - \sum_{i \in N \setminus S} x_i, & \text{for } T = S, \\ \max_{Q \subset N \setminus S}(v(T \cup Q) - \sum_{i \in Q} x_i) & \text{otherwise .} \end{cases} \tag{2}$$

If for some universe $\mathcal{N}$ we consider the set of all consistent solutions, then in this set the single-valued solutions turn out to be *minimal* under inclusion solutions, and those satisfying more converse consistency are *maximum* under inclusion solutions. Among the set of covariant and consistent solutions, satisfying the equal treatment property, there are the unique single-valued one – it is the prenucleolus (Sobolev, 1975), and the maximum one – it is the prekernel (Peleg, 1986).

Recall their definitions.

Given a TU game (N, v), its payoff vector $x \in X(N, v)$, and a coalition $S \subsetneqq N$, the *excess* of a coalition S with respect to x is equal to $v(S) - x(S)$. This difference is a total amount that the coalition S will have after paying x_i to each player $i \in S$. By $e(x) = \{e(S, x)\}_{S \subset N}$ we denote the *excess vector* of x.

Denote by $\theta(x) \in \mathbb{R}^{2^N}$ the vector whose components coincide with those of $e(x)$, but arranged in a weakly decreasing manner, that is,

$$\theta^t(x) := \max_{\substack{\mathcal{T} \subset 2^N \\ |\mathcal{T}|=t}} \min_{S \in \mathcal{T}} e(S,x) \;\; \forall t = 1, ..., p, \text{ whereas } p := 2^{|N|} - 2.$$

We will use also the notation $\theta_v(x)$ if it is necessary to indicate the characteristic function in the definition of the excess vector.

By $e^k(x)$ we will denote the k-valued component of the vector $\theta(x)$ such that $e^1(x) > e^2(x) > ...e^k(x)$ for some k, and by

$$\mathcal{S}_j(v,x) = \{S \subsetneqq N \,|\, v(S) - x(S) = e^j(x)\} \tag{3}$$

the set of coalitions on which the j-valued excess of the vector x is attained.

Let $\geq_{lex}$ denote the lexicographic order in $\mathbb{R}^m$ for an arbitrary m :

$$x \geq_{lex} y \iff x = y \text{ or } \exists 1 \leq k \leq m \text{ such that } x_k > y_k \text{ and } x_i = y_i \text{ for } i < k.$$

The *prenucleolus* of the game (N,v), $PN(N,v)$, is the unique efficient payoff vector such that

$$\theta(x) \geq_{lex} \theta(PN(N,v)) \text{ for all } x \in X^*(N,v). \tag{4}$$

The existence and the uniqueness of the prenucleolus for each TU game follows from Schmeidler's theorem (Schmeidler, 1969) though he considered the *nucleolus* defined as in (4) only for all *individual rational payoff vectors (imputations)*: $x \in I(N,v)$, where

$$I(N,v) = \{x \in X^*(N,v) \,|\, x_i \geq v(\{i\}) \;\; \forall i \in N\}.$$

For each $i, j \in N$ and a payoff vector x the *maximum surplus* of i over j in x is denoted by

$$s_{ij}(x,v) = \max_{S \ni i, S \not\ni j} (v(S) - x(S)).$$

The *prekernel* of a game (N,v), $PK(N,v)$, is the set

$$PK(N,v) = \{x \in X^*(N,v) \,|\, s_{ij}(x,v) = s_{ji}(x,v) \text{ for all } i,j \in N\}. \tag{5}$$

The prenucleolus and the prekernel have the following axiomatic characterizations.

Theorem 1 ((Sobolev, 1975)). *If $\mathcal{N}$ is infinite, then the unique solution satisfying SV, COV, ANO, and CONS is the prenucleolus.*

Theorem 2 ((Orshan, 1993)). *If $\mathcal{N}$ is infinite, then the unique solution satisfying SV, COV, ETP, and CONS is the prenucleolus.*

Theorem 3 ((Peleg, 1986)). *For an arbitrary set $\mathcal{N}$ the unique solution satisfying NE, PO, COV, ETP, CONS, and CCONS is the prekernel.*

3. The least core and its consistent subsolutions

The *least core (LC)* of a game (N, v) is defined by

$$LC(N, v) = \arg \min_{x \in X^*(N,v)} \max_{S \subsetneqq N} (v(S) - x(S)).$$

The least core is non-empty for all TU games, it is efficient, anonymous, and covariant. However, it is not consistent.

By the definition it follows that $x \in LC(N, v)$ implies that $\max_{S \subset N}(v(S) - x(S)) = e^1(x)$, where $e^1(x)$ is the maximal components of the excess vector $e(x)$. Thus, the maximal components of the excess vectors $e(x)$ for $x \in LC(N, v)$ and $e(PN(N, v))$ coincide.

Let σ be an arbitrary consistent subsolution of the least core. Then for every two-person game (N, v) $\sigma(N, v) = PK(N, v) = LC(N, v)$, and, by consistency of σ and the cited Theorem 3, $\sigma(N, v) \subset PK(N, v)$ for every TU game (N, v).

Evidently, the class Σ of all consistent subsolutions of the least core is closed under the union: if $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \vee \sigma_2 \in \Sigma$, where for every game (N, v) $(\sigma_1 \vee \sigma_2)(N, v) = \sigma_1(N, v) \cup \sigma_2(N, v)$

Let us consider the solution $\sigma^* = \vee_{\sigma \in \Sigma}$. Then the solution $\sigma^* \in \Sigma$, and it is the *maximum* under inclusion solution from the class Σ.

Recall the definition of balancedness.

A collection $\mathcal{B}$ of coalitions from the set N is *balanced*, if there exists positive numbers λ_S for all $S \in \mathcal{B}$ such that

$$\sum_{\substack{S \ni i \\ S \in \mathcal{B}}} \lambda_S = 1 \tag{6}$$

for all $i \in N$. It is balanced on $T \subset N$, if equalities (6) hold only for $i \in T$.

A collection $\mathcal{B}$ is *weakly balanced* if it contains a balanced subcollection.

We begin to characterize the solution σ^* with the recurrent in the number of players formula.

It is clear that for two-person games, $|N| = 2$, $\sigma^*(N, v) = LC(N, v) = PK(N, v)$ is the *standard* solution.

It is known that for $x \in LC(N, v)$ the collection $\mathcal{S}_1(v, x)$ (3) is weakly balanced, hence, it contains a balanced subcollection. Let $\mathcal{S}_1(v)$ be such a balanced subcollection

$$\mathcal{S}_1(v) = \bigcap_{x \in LC(N,c)} \mathcal{S}_1(v, x). \tag{7}$$

It generates a partition $\mathbf{T}_1(v) = \{T_1, ..., T_k\}$ of the set N defined by:
for every $j = 1, ..., k$ and $S \in \mathcal{S}_1(v)$ T_j is the maximal under inclusion subset of players such that either $T_j \subset S$, or $T_j \cap S = \emptyset$. In accordance with the paper (Maschler, Peleg and Shapley, 1979), we call the partition $\mathbf{T}_1$ *the partition induced by the collection* $\mathcal{S}_1(v)$.

A collection $\mathcal{S}$ of coalitions from N is called *separating*, if $S \in \mathcal{S}$, $i \in S$, $j \notin S$, $i, j \in N$ implies the existence $T \in \mathcal{S}$ such that $j \in T, i \notin T$.

If for a separating collection of coalitions $\mathcal{S}$ the induced partition $\mathbf{T}(\mathcal{S})$ is a partition on singletons, then we call such a collection *completely separating*.

Proposition 1. *A solution σ belongs to the class Σ : $\sigma \in \Sigma$ if and only if for every game (N, v)*

$$\sigma(N,v) = \{x \in LC(N,v) \,|\, x_{T_j} \in \sigma(T_j, v^x_{T_j}), \quad j = 1, ..., k\}, \tag{8}$$

where $(T_j, v^x_{T_j})$ is the reduced game of (N,v) on the player set T_j and with respect to x.

Proof. It is clear that every vector $x \in \sigma(N,v)$ satisfies the right-hand part of equality (8).

Let us show the inverse inclusion. Consider an arbitrary vector x satisfying the right-hand part of equality (8). Then by the definition of the partition $\mathbf{T}_1(v)$ for players $k \in T_i, l \in T_j, i \neq j$ it holds the equality

$$s_{kl}(x) = s_{lk}(x). \tag{9}$$

If both players $k, l \in T_j$, then equality (9) follows from consistency of σ implying $x_{T_j} \in \sigma(T_j, v^x_{T_j})$. Hence, $x \in PK(N,v)$.

Let us show that for every coalition $S \subset N$ $x_S \in LC(S, v^x_S)$.

Consider the following cases:

1) $S \subset T_j$ for some $j = 1, ..., k$. Path independence property of the reduced characteristic functions (Pechersky and Yanovakaya, 2004) implies equality

$$v^x_S = (v^x_{T_j})^x_S,$$

from which and from $x_{T_j} \in \sigma(T_j, v^x_{T_j})$ by the inductive assumption it follows $x_S \in \sigma(S, v^x_S)$, $x_S \in LC(S, v^x)$.

2) $\exists i, j = 1, ..., k \quad S \cap T_i \neq \emptyset, S \cap T_j \neq \emptyset$. Without loss of generality assume that $S \subset T_i \cap T_j$. Then in the reduced game (S, v^x) $\mathcal{S}_1(v^x_S) = \mathcal{S}_1(v)|_{x_{N \setminus S}} \neq \{S\}$.

Therefore, $x_S \in LC(S, v^x)$, and for every reduced game (S, v^x_S) of the game (N,v) every vector x, satisfying the right-hand part of equality (8), belongs to the set $(LC \cap PK)(S, v^x_S)$. Since $x \in (LC \cap PK)(N,v)$, this means that the solution defined in the right-hand part of equality (5), is a consistent subsolution of the solution $(LC \cap PK)$.

The following example shows that σ^* is a proper subsolution of the solution $(LC \cap PK)$.

Example 1. Consider a five-person game being a version of the known game from (Davis and Maschler, 1965)
$N = \{1, 2, 3, 4, 5\}$, $v(N) = 7, v(\{i, j, k\}) = 3$ for all $i, j = 1, 2, 3,\ k = 4, 5$,
$v(S) = 0$ for other $S \subset N$.

Since the players 1,2,3 and 4,5 are substitutes, $PK(N,v) = \{t, t, t, \frac{7-3t}{2}, \frac{7-3t}{2}\}_{t \in [3/2, 1]}$, and $(LC \cap PK)(N,v) = PN(N,v) = (\frac{3}{2}, \frac{3}{2}\frac{3}{2}, \frac{5}{4}, \frac{5}{4}) = x^*$, $\max_{ij} s_{ij}(x) = -5/4$. This maximum is attained on coalitions (i, j, k), for $i, j = 1, 2, 3,\ k = 4, 5$, and on singletons $k \in \{4, 5\}$.

Replicate this game. Let $N^* = \{6, 7, 8, 9, 10\}$, and consider the game $(N \cup N^*, \tilde{v})$, where $\tilde{v}(N \cup N^*) = 14, \tilde{v}(S) = v(S)$ for $S \subset N \quad S \subset N^*$, and for

$S = P \cup Q, P \subset N, Q \subset N^*$ $\tilde{v}(S) = \tilde{v}(P) + \tilde{v}(Q)$. Then

$$LC(N \cup N^*, \tilde{v}) = \{(x, y) \,|\, x, y \in \mathbb{R}^5_+, \}, \text{ where } \sum_{i=1}^{5} x_i = \sum_{j=6}^{10} y_j = 7, \;\; x_i, y_j \geq 0.$$

Thus,

$$(LC \cap PK)(N \cup N^*, \tilde{v}) = \{(x, y) \,|\, x \in PK(N, v), y \in PK(N^*, \tilde{v})\}.$$

Now let us consider the reduced game (N, v^y) of $(N \cup N^*, \tilde{v})$ on the player set N with respect to an arbitrary vector $(x, y) \in PK(N \cup N^*, \tilde{v})$, and let $x \neq x^*$. The definition of the reduced game implies that $v^y = v$. Therefore, in the reduced game $x \notin (LC \cap PK)(N, v^y) = (LC \cap PK)(N, v)$.

4. The maximum consistent subsolution of the least core

In this section we give a combinatorial characterization, firstly applied by Kohlberg (Kohlberg, 1971) for the characterization of the prenucleolus, of the maximum consistent subsolution of the least core.

Let us recall formula (8). Its right-hand part consists of the solutions of the reduced games on the coalitions of the partition of the set of players, induced by the collection $\mathcal{S}_1(v)$ of coalitions on which the minimum of the maximal excesses for all vectors from the least core is attained.

Note that by definition (2) the reduced game (S, v^x) of a game (N, v) on an arbitrary set S of players and with respect to x does not depend on characteristic function values $v(T)$ for $T \subsetneqq N \setminus S$. Therefore, all reduced games on coalitions of an arbitrary partition $\mathbf{T}$ of the set of players do not depend on the values $v(S)$ for $S \in \mathcal{T}$, where $\mathcal{T}$ is the collection of all coalitions being unions of the coalitions of the partition $\mathbf{T}$.

Thus, when the least core of a game (N, v) has been defined, the characteristic function values of coalitions being unions of coalitions of the partition $\mathbf{T}_1(v)$, turn out to be inessential for the definition $\sigma(N, v)$ of arbitrary solution $\sigma \in \varSigma(N, v)$. Just this fact is a basic tool for the following description of the set of preimputations belonging to any consistent subsolution, including the maximum one, of the least core.

First, introduce the following notation. Given an arbitrary game (N, v) an a preimputation $x \in X^*(N, v)$ let

$\mathbf{T}_1(v, x)$ be the partition of N, induced by the collection $\mathcal{S}_1(v, x)$ that was defined in (5);

$\mathbf{T}_0(v, x) = \{N\}$;

$I_1(v, x) = \{(i, j) \,|\, s_{ij}(x) = \max_{i', j' \in N} s_{i'j'}(x)\}$ the set of pairs of players on which the largest maximal surplus value for x is attained;

for $h > 1$ $I_h(v, x) = \{(i, j) \,|\, s_{ij}(x) = \max\limits_{(i', j') \notin \bigcup_{l=1}^{h-1} I_l(x)} s_{i'j'}(x)\}$ is the set of pairs of players on which the h-th maximal surplus value is attained;

$\mathcal{S}^{ij}(v,x) = \arg\max_{\substack{S \ni i \\ S \not\ni j}}(v(S) - x(S))$ the collection of coalitions on which the maximal surplus value for x equals $s_{ij}(x)$;

$$\mathcal{E}_k(v,x) = \bigcup_{(i,j)\in I_k(v,x)} \mathcal{S}^{ij}(v,x) \tag{10}$$

the collection of all coalitions on which the k-th maximal surplus values are attained. Evidently, $\mathcal{E}_1(v,x) = \mathcal{S}_1(v,x)$;

$\mathbf{T}_k(v,x)$ is the partition of N, induced by the collection $\bigcup_{h=1}^k \mathcal{E}_h(v,x)$.

Arrange the values $s_{ij}(x)$ in a decreasing manner:

$$\begin{gathered} s^1(x) = s_{ij}(x),\ (i,j) \in I_1(v,x), \\ \dots\ \dots \\ s^l(x) = s_{ij}(x),\ (i,j) \in I_l(v,x), \end{gathered}$$

where $s^l(x)$ is the minimal value of maximal surpluses for x, (generally, $l = l(x)$).

It s known that the values $s_{ij}(x)$ do not change under reducing, that is for every coalition $T \subset N$ and players $i,j \in T$ the equality $s_{ij}(x) = s_{ij}(x_T)$ holds, where $s_{ij}(x_T)$ is the maximal surplus value of player i over the player j in the reduced game (T, v^x) with respect to x. Hence, for this reduced game the following equalities hold as well

$\mathcal{S}^{ij}(v,x)|_T = \mathcal{S}^{ij}(v^x, x_T) \quad i,j \in T;$

$$\mathcal{E}_k(v,x)|_T = \begin{cases} T, & \text{if } T \subset S\, \forall S \in \mathcal{E}_k(v,x), \\ \mathcal{E}_h(v^x,x_T),\ \text{for some } h \le k & \text{otherwise}, \end{cases} .$$

Hence, the excess values $v(S) - x(S) \in (s^h(x), s^{h-1}(x))$ for $h = 2,...,l$ and for coalitions $S \notin \bigcup_{h=1}^l \mathcal{E}_h(v,x)$ have no influence on the similar values in the reduced games.

Theorem 4. *Given a game (N,v), its preimputation $x \in \sigma^*(N,v)$ if and only if the collections $\mathcal{E}_h(v,x)$ are balanced on all sets $T \in \mathbf{T}_{h-1}(v,x), |T| > 1$, $h = 1,...,l(x)$.*

Proof. The 'only if' part. Let $x \in \sigma^*(N,v)$. For $|N| = 2$ the conditions of the Theorem is fulfilled, since for two-person games σ^* is the standard solution.

Assume that the condition of the Theorem for x have been fulfilled for all TU games whose numbers of of players are less than $n = |N|$.

Let a collection $\mathcal{E}_h(v,x)$ is not balanced on some $T \in \mathbf{T}_{h-1}(v,x)$, $|T| > 1$ and for some $1 < h < l(x)$. Then in the reduced game (T, v^x) the collection $\mathcal{E}_f(v^x, x_T) = \mathcal{E}_h(v,x)|_T$ for some $f = f(h) \le h$, and it is not balanced. Since σ^* is a consistent solution, we should have the inclusion $x_T \in \sigma(T, v^x)$, that means that for the reduced game (T, v^x) the conditions of the Theorem violates, and we obtain a contradiction.

The 'if' part.

Let (N,v) be an arbitrary game and a preimputation x satisfies all conditions of the Theorem. Then the collections $\mathcal{E}_h(v,x)$ are balanced on every two-person coalition for all $h = 1,...,l$. In fact, for every $h = 1,...,l$ any players $k,m \in N$ either belong to a coalition $T \in \mathbf{T}_{h-1}(v,x)$ on which the collection $\mathcal{E}_h(v,x)$ is balanced, or they belong to different coalitions from the partition $\mathbf{T}_h(v,x)$ and, hence, they are separated.

It is clear that for $|N| = 2$ balancedness of the collection $\mathcal{S}_1(v,x) = \mathcal{E}_1(v,x)$ implies that $\sigma^*(N,v) = LC(N,v) = PK(N,v)$ is the standard solution.

Let now $|N| > 2$ and assume that the 'if' part of the Theorem holds for all games with the number of players less than $n = |N|$. The collection $\bigcup_{h=1}^{l(x)} \mathcal{E}_h(v,x)$ is completely separating, Since every balanced collection is separating, balancedness of the collections $\mathcal{E}_h(v,x)$ on $T \in \mathcal{T}_{h-1}(v,x)$, $h = 1, ..., l(x)$ implies that $x \in PK(N,v)$.

Equality $\mathcal{E}_1(v,x) = \mathcal{S}_1(v,x)$ and balancedness of this collection on N yields $x \in LC(N,v)$. Thus, it only remains to show consistency, i.e. that for every coalition $T \subset N$ equality (8) holds

$$x_T \in LC(T, v^x) \tag{11}$$

The definition of the collection $\mathcal{E}_1(v,x) = \mathcal{S}_1(N,v)$ yields that for every coalition $T \not\subset T' \in \mathbf{T}_1(v,x))$ inclusion (11) holds.

Let us consider the reduced game (T, v^x), where $T \in T' \in \mathbf{T}_1$. Let $h > 1$ ne the minimal number for which $\mathcal{E}_j(v,x)|_T \neq \{T\}$. This means that $\mathcal{E}_1(v^x, x_T) = \mathcal{E}_j(v,x)|_T$. The collection $\mathcal{E}_j(v,x)$ is balanced on every coalition $T' \in \mathbf{T}_{j-1}(v,x)$, and the set $T \subset T'$ itself is one of such a T', since for $f < h$ $\mathcal{S}_f|_T = T$ by the definition of h. Therefore, $x_T \in LC(T, v^x)$.

5. An axiomatic characterization of the maximum consistent subsolution of the least core.

Denote by $\mathcal{G}^b \subset \mathcal{G}_\mathcal{N}$ the class of all *balanced* games , i.e., the class of games with nonempty cores, and by $\mathcal{G}^{tb} \subset \mathcal{G}^b$ the class of all *totally balanced* games, i.e. games whose every subgame is balanced.

Peleg (Peleg, 1986) gave the following axiomatic characterizations of the core (C) and of the intersection of the core $(C \cap PK)$ with the prekernel of the class of totally balanced games.

Theorem 5 (Peleg, 1986). *The solution $C \cap PK$ is a unique solution for the class $\mathcal{G}^{tb}$, satisfying efficiency, individual rationality, equal treatment property, weak consistency, and converse consistency.*

This characterization holds for the class of balanced games $\mathcal{G}^b$ as well. Moreover, converse consistency can be replaced by maximality under inclusion.

Let us compare the solution $C \cap PK$ with the maximum consistent subsolution of the least core σ^*. The last solution is defined in the whole class $\mathcal{G}_\mathcal{N}$ and satisfies all the axioms of Peleg's theorem except for individual rationality. It turns out that these axioms together with shift invariance are sufficient for the axiomatic characterization of the solution σ^*.

Theorem 6. *The solution σ^* is the unique maximal under inclusion solution for the class $\mathcal{G}_\mathcal{N}$, satisfying axioms non-emptiness, efficiency, equal treatment property, covariance, shift invariance, and consistency in the class $\mathcal{G}^b$ of balanced games.*

Proof. It is clear that the solution σ^* satisfies all these axioms in the whole class $\mathcal{G}_\mathcal{N}$. Thus, it suffices to check consistency in the class of balanced games. Thus, we should show that for every balanced game (N,v), every $x \in \sigma^*(N,v)$, and for a coalition $S \subset N$, the reduced game is balanced. The least core of every balanced game is contained in its core, hence, $x \in C(N, v)$. It is known that the core is consistent in the class of balanced games (Peleg, 1986). Therefore, every reduced game of the game (N, v) with respect to any vector from the core, including x, is balanced.

Before the proof of uniqueness let us note that for every game (N, v) and for its reduced game (S, v^x) on a coalition S with respect to x, the $v^x - a = (v - a)^x$, holds for all numbers a, where

$$(v-a)(T) = \begin{cases} v(N), & \text{if } T = N, \\ v(T) - a & \text{for other } T \subset N. \end{cases}$$

Let now σ be an arbitrary solution satisfying all the axioms stated in the Theorem. Let (N, v) be an arbitrary game, and a be the number such that $C(N, v - a) = LC(N, v - a) = LC(N, v)$. Let $x \in \sigma(N, v)$ be an arbitrary vector. By shift invariance of σ $\sigma(N, v) = \sigma(N, v - a)$. By consistency of σ in the class of balanced games for every coalition S $x_S \in \sigma(S, (v-a)^x) = \sigma(S, v^x - a)$. From shift invariance of σ it follows $x_s \in \sigma(S, v^x)$, that means that the solution σ is consistent on the whole class $\mathcal{G}_{\mathcal{N}}$. Axioms efficiency, equal treatment property, covariance and consistency of σ on the class $\mathcal{G}_{\mathcal{N}}$ implies that $\sigma \subset PK$ by the cited Theorem 5.

Let us show that $\sigma \subset LC$. For two-person games we have $\sigma = PK = LC$. Assume that there is a vector $x \in \sigma(N, v)$ such that $x \notin LC(N, v) = C(N, v - a)$. Then by shift invariance of σ $x \in \sigma(N, v - a)$, and by consistency and converse consistency of the core (Peleg, 1986) there are players $i, j \in N$ such that the reduce game $\{i, j\}, (v - a)^x)$ is not balanced that contradicts consistency of σ in the class of balanced games.

Thus, we have obtained that every solution σ satisfying the axioms stated in the Theorem, is a consistent subsolution of the least core.Therefore, the maximum of such solutions is the solution σ^*.

On the contrary to Peleg's theorem, the maximality axiom in Theorem 5 can be replaced by converse consistency. In fact, it is clear that the solution σ^* satisfies converse consistency on the class of balanced games. It follows from the coincidence of σ^* with the standard solution on the class of two-person games, shift invariance and converse consistency of the core and of the prekernel on the class of balanced games. Converse consistency, in its turn, implies maximality under inclusion on the class of balanced games. At last, shift invariance spreads maximality on the class of all games.

6. Examples

Let us give an example of consistent subsolutions of the least core. Define a solution τ on the class $\mathcal{G}_{\mathcal{N}}$ as follows: for every game (N, v)

$$\tau(N, v) = \sigma_{k(N,v)}(N, v), \tag{12}$$

where $k(N, v)$ is the minimal number for which the collection $\bigcup_{j=1}^{j(N,v)} \mathcal{S}_j(v)$ is completely separating.

Evidently, $\tau \subset (LC \cap PK)$. It is easy to check that the solution τ is consistent, and, hence, $\tau \subset \sigma^*$.

Let us show that τ is a proper subsolution of σ^*.

Example 2. $N = \{1,2,3,4,5,6,7,8\}, v(N) = 10, v(1,3,5,7) = v(2,4,6,8) = 4,$ $v(1,2) = v(3,4) = v(5,6) = v(7,8) = 1, v(\{i\}) = -1/2, i = 1,...,8,\ \ v(S) = 0$, for other coalitions.

For this game

$$\mathcal{S}_1(v) = \{1,3,5,7\}, \{2,4,6,8\},$$

$$LC(N,v) = \{x \,|\, x_i \geq 1/2, x_1 + x_3 + x_5 + x_7 = x_2 + x_4 + x_6 + x_8 = 5\}, \tag{13}$$

$e^1(x) = e_{max} = -1$ for all $x \in LC(N,v)$.

The second minimization of the ordered excess vector yields the second value excess $e_2 = -\frac{3}{2}$, that for all $x \in \tau(N,v)$ is attained on coalitions

$$\mathcal{S}_2(v) = (1,2), (3,4), (5,6), (7,8), \tag{14}$$

and, possibly, on some others, because $\mathcal{S}_2(v,x) \supset \mathcal{S}_2(v)$.

Equality (14) yields that for $x \in \tau(N,v)$

$$x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = \frac{5}{2}. \tag{15}$$

The collection $\mathcal{S}_1(v) \cup \mathcal{S}_2(v)$ is completely separating, hence,

$$\tau(N,v) = \alpha\left(\frac{1}{2},2,2,\frac{1}{2}\right) + (1-\alpha)\left(2,\frac{1}{2},\frac{1}{2},2\right), \beta\left(2,\frac{1}{2},\frac{1}{2},2\right) + (1-\beta)\left(2,\frac{1}{2},\frac{1}{2},2\right) \tag{16}$$

for any $\alpha, \beta \in [0,1]$. Here components with α have coordinates from 1 to 4, and components with β have coordinates from 5 to 8.

In this example $\tau(N,v) = \sigma^*(N,v) = LC \cap PK$, since in the least core the players $i, i+1(\text{mod}(8)$, $i \in N$ are separated, and for players $i, i+2(\text{mod}8), i \in N$ the largest maximal surplus values $s_{ii+2}(x)$ are attained on the coalitions $\{i, i+1\}$, implying equalities (15). Therefore, the solution $\tau(N,v)$ coincides with $\sigma^*(N,v)$, which, in its turn, coincides with $LC \cap PK$.

Note that there are two permutations $\pi_k : N \to N$, $\pi_1(i) = i+1(\text{mod}8)$, $\pi_2(i) = i+2(\text{mod}8)$ such that $\pi_k(N,v) = (N,v), k = 1,2$. By anonymity of the prenucleolus we should have $PN(N,v) = (\frac{v(N)}{8},...,\frac{v(N)}{8}) = \left(\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4}\right)$.

This vector can be also obtained after the third minimization of the excess vector, giving the third excess value $e_3 = -\frac{7}{4}$, attained on the singletons.

Example 3. Consider a slight modification (N',v') of the game (N,v) from Example 2, where $N' = \{1',2',3',4',5'.6',7',8'\}$, and whose characteristic function is defined as follows:

$$v'(S') = \begin{cases} 3/2, & \text{for coalitions } \{1',2'\}, \{3',4'\}, \{5',6'\}, \{7',8'\}, \\ v'(S') & \text{for other coalitions } S' \subset N. \end{cases}$$

For this game we obtain that the collection
$\mathcal{S}_1(v') = \{\{1',3',5',7'\}, \{2',4',6',8'\}, \{1',2'\}, \{3',4'\}, \{5',6\}, \{7',8'\}\}$
is already completely separating, hence, the solutions $LC(N,v') = \sigma^*(N,v') = \tau(N,v')$ are defined by equality (13) with replacing i on i', $i,i' = 1,...,8$.

Define a composition $(N \cup N', w)$ of the games (N, v) and (N', v'), whose characteristic function w is additive with respect to v and v' : for every $Q \subset N \cup N'$

$$w(Q) = v(S) + v'(T), \text{ if } Q = S \cup T, S \subset N, T \subset N'.$$

Then

$$\mathcal{S}_1(w) = \{1, ..., 8\}, \{1', ..., 8'\} \text{ implying } e^1(x) = e_{\max}(w) = 0 \tag{17}$$

for all $x \in LV(N \cup N', w)$. Since the game $(N \cup N', w)$ is balanced, its reduced games on the sets N and N' with respect to vectors from the core (including the least core) coincide with the games (N, v) and (N', v') respectively.

Therefore, the solution set $\sigma^*(N \cup N^*, w)$ is equal to product

$$\sigma^*(N \cup N^*, w) = \sigma^*(N, v) \times \sigma^*(N', v'). \tag{18}$$

Let us find the solution set $\tau(N \cup N', w)$. Evidently, $\mathcal{S}_2(w) = \mathcal{S}_1(v) \cup \mathcal{S}_1(v')$, and the collections $\mathcal{S}_1(w)$ (17) and $(\mathcal{S}_1 \cup \mathcal{S}_2)(w)$ are not completely saparating, since collection $\mathcal{S}_1(v)$ (13) is not completely separating on N.

The collection $\mathcal{S}_3(w)$ is equal to

$$\mathcal{S}_3(w) = \mathcal{S}_2(v) \times \mathcal{S}_2(v').$$

The collection $\mathcal{S}_2(v)$ is defined in (14), and the collection $\mathcal{S}_2(v')$ consists of singletons $\{1'\}, ...\{8'\}$.

Therefore, $e_2(v') = -\frac{7}{4}$, the collection $\mathcal{S}_1(v') \cup \mathcal{S}_2(v')$ is completely separating, and we obtain

$$\sigma^*(N', v') = \tau(N', v') = PN(N', v') = \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right).$$

Thus, for the game $(N \cup N', w)$

$$\tau(N \cup N', w) = \tau(N, v) \times \left(\frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}, \frac{5}{4}\right) \subsetneq \sigma^*(N \cup N', w).$$

7. Conclusion

The results given in the paper is a step to the description of all TU game solutions satisfying efficiency, anonymity/equal treatment property, covariance, and Davis–Maschler consistency. At present, apart from the solutions presented here, only the k-prekernels are known (Katsev and Yanovskaya, 2009). All they are contained in the prekernel, but, at the same time, have some "nucleolus-type" traits connected with lexicographic optimization of excess and of maximum surplus vectors. The open problem is to find other such solutions.

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Subgame Consistent Cooperative Solutions in Stochastic Differential Games with Asynchronous Horizons and Uncertain Types of Players

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Abstract This paper considers cooperative stochastic differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty. Subgame consistent cooperative solutions and analytically tractable payoff distribution mechanisms leading to the realization of these solutions are derived. This analysis widens the application of cooperative stochastic differential game theory to problems where the players' game horizons are asynchronous and the types of future players are uncertain. It represents the first time that subgame consistent solutions for cooperative stochastic differential games with asynchronous players' horizons and uncertain types of future players are formulated.

Keywords: Cooperative stochastic differential games, subgame consistency, asynchronous horizons, payment distribution mechanism.

AMS Subject Classifications. Primary 91A12; Secondary 91A25.

1. Introduction

In many game situations, the players' time horizons differ. This may arise from different life spans, different entry and exit times in different markets, and the different duration for leases and contracts. Asynchronous horizon game situations occur frequently in economic and social activities. Moreover, only the probability distribution of the types of future players may be known. In this paper, we consider cooperative stochastic differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty.

Cooperative games suggest the possibility of socially optimal and group efficient solutions to decision problems involving strategic action. In dynamic cooperative games, a stringent condition for a dynamically stable solution is required: In the solution, the optimality principle must remain optimal throughout the game, at any instant of time along the optimal state trajectory determined at the outset. This condition is known as *dynamic stability or time consistency.* The question of dynamic stability in differential games has been rigorously explored in the past three decades. (see Haurie (1976), Petrosyan and Danilov (1982) and Petrosyan (1997)). In the presence of stochastic elements, a more stringent condition – that of *subgame consistency* – is required for a dynamically stable cooperative solution. In particular, a cooperative solution is subgame-consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behavior would remain optimal. In particular dynamic consistency ensures that as the game proceeds players are guided by the same optimality principle at each instant of time, and hence do not possess incentives to deviate from the previously adopted optimal behavior. A rigorous framework for the study of subgame-consistent solutions in cooperative stochastic differential games was established in the work of (Yeung and Petrosyan (2004, 2005 and 2006). A generalized theorem was developed for the derivation of an analytically tractable "payoff distribution procedure" leading to subgame consistent solutions.

In this paper, subgame consistent cooperative solutions are derived for stochastic differential games with asynchronous players' horizons and uncertain types of future players. Analytically tractable payoff distribution mechanisms which lead to the realization of these solutions are derived. This analysis extends the application of cooperative stochastic differential game theory to problems where the players' game horizons are asynchronous and the types of future players are uncertain. The organization of the paper is as follows. Section 2 presents the game formulation and characterizes noncooperative outcomes. Dynamic cooperation among players coexisting in the same duration is examined in Section 3. Section 4 provides an analysis on payoff distribution procedures leading to subgame consistent solutions in this asynchronous horizons scenario. An illustration in cooperative resource extraction is given in Section 5. Concluding remarks and model extensions are given in Section 6.

2. Game Formulation and Noncooperative Outcome

In this section we first present an analytical framework of stochastic differential games with asynchronous players' horizons, and characterize its noncooperative outcome.

2.1. Game Formulation

For clarity in exposition and without loss of generality, we consider a general class of stochastic differential games, in which there are $\upsilon+1$ overlapping cohorts or generations of players. The game begins at time t_1 and terminates at time $t_{\upsilon+1}$. In the time interval $[t_1, t_2)$, there coexist a generation 0 player whose game horizon is $[t_1, t_2)$ and a generation 1 player whose game horizon is $[t_1, t_3)$. In the time interval $[t_k, t_{k+1})$ for $k \in \{2, 3, \cdots, \upsilon-1\}$, there coexist a generation $k-1$ player whose game horizon is $[t_{k-1}, t_{k+1})$ and a generation k player whose game horizon is $[t_k, t_{k+2})$. In the last time interval $[t_\upsilon, t_{\upsilon+1}]$, there coexist a generation $\upsilon - 1$ player and a generation υ player whose game horizon is just $[t_\upsilon, t_{\upsilon+1}]$.

For the sake of notational convenience in exposition, the player who enters the game at time t_k can be of types $\omega_{a_k} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$. When the game starts at initial time t_1, it is known that in the time interval $[t_1, t_2)$, there coexist a type ω_1 generation 0 player and a type ω_2 generation 1 player. At time t_1, it is also known that the probability of the generation k player being type $\omega_{a_k} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$ is $\lambda_{a_k} \in \{\lambda_1, \lambda_2, \cdots, \lambda_{\varsigma_k}\}$, for $k \in \{2, 3, \cdots, \upsilon\}$. The type of generation k player will become known with certainty at time t_k.

The instantaneous payoff functions and terminal rewards of the type ω_{a_k} generation k player and the type $\omega_{a_{k-1}}$ generation $k-1$ player coexisting in the time interval $[t_k, t_{k+1})$ are respectively:

$$g^{k-1(\omega_{k-1})}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \quad \text{and} \quad q^{k-1(\omega_{k-1})}[t_{k+1},x(t_{k+1})],$$

$$\text{and} \quad g^{k(\omega_k)}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \quad \text{and} \quad q^{k(\omega_k)}[t_{k+2},x(t_{k+2})], \tag{2.1}$$

for $k \in \{1,2,3,\cdots,\upsilon\}$,

where $u_{k-1}^{(\omega_{k-1},O)\omega_k}(s)$ is the vector of controls of the type $\omega_{a_{k-1}}$ generation $k-1$ player when he is in his last (old) life stage while the type ω_{a_k} generation k player is coexisting;

and $u_k^{(\omega_k,Y)\omega_{k-1}}(s)$ is that of the type ω_{a_k} generation k player when he is in his first (young) life stage while the type $\omega_{a_{k-1}}$ generation $k-1$ player is coexisting.

Note that the superindex "O" in $u_{k-1}^{(\omega_{k-1},O)\omega_k}(s)$ denote Old and the superindex "Y" in $u_k^{(\omega_k,Y)\omega_{k-1}}(s)$ denote young. The state dynamics of the game is characterized by the vector-valued stochastic differential equations:

$$\frac{dx(s)}{ds} = f[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]ds+\sigma[s,x(s)]dz(s), x(t_1)=x_0 \in X, \tag{2.2}$$

for $s \in [t_k, t_{k+1})$,

if the type ω_{a_k} generation k player and the type $\omega_{a_{k-1}}$ generation a_{k-1} player coexisting in the time interval $[t_k,t_{k+1})$ for $k \in \{1,2,3,\cdots,\upsilon\}$, and where $\sigma[s,\ x(s)]$ is a n $\times$ Θ matrix and $z(s)$ is a Θ-dimensional Wiener process. Let $\Omega[s,x(s)] = \sigma[s,x(s)]\sigma[s,x(s)]'$ denote the covariance matrix with its element in row h and column ζ denoted by $\Omega^{h\zeta}[s,x(s)]$.

In the game interval $[t_k,t_{k+1})$ for $k \in \{1,2,3,\cdots,\upsilon-1\}$ with type ω_{k-1} generation $k-1$ player and type ω_k generation k player is of, the type ω_{k-1} generation $k-1$ player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k-1(\omega_{k-1})}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]\,e^{-r(s-t_k)}ds \right.$$
$$\left. +e^{-r(t_{k+1}-t_k)}q^{k-1(\omega_{k-1})}[t_{k+1},x(t_{k+1})] \,\middle|\, x(t_k)=x \in X \right\} \tag{2.3}$$

and the type ω_k generation k player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k(\omega_k)}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]e^{-r(s-t_k)}ds \right.$$

$$+\sum_{\alpha=1}^{\varsigma} \lambda_{a_{k+1}} \int_{t_{k+1}}^{t_{k+2}} g^{k(\omega_k)}[s,x(s),u_k^{(\omega_k,O)\omega_\alpha}(s),u_{k+1}^{(\omega_\alpha,Y)\omega_k}(s)]e^{-r(s-t_k)}ds$$

$$\left. +e^{-r(t_{k+2}-t_k)}q^{k(\omega_k)}[t_{k+2},x(t_{k+2})] \,\middle|\, x(t_k)=x \in X \right\} \tag{2.4}$$

subject to stochastic dynamics

$$\frac{dx(s)}{ds} = f[s, x(s), u_{h-1}^{(\omega_{h-1},O)\omega_h}(s), u_h^{(\omega_h,Y)\omega_{h-1}}(s)]ds + \sigma[s, x(s)]dz(s), x(t_k) = x,$$

for $s \in [t_h, t_{h+1})$ and $h \in \{k, k+1, \cdots, \upsilon\}$,

where r is the discount rate.

In the last time interval $[t_\upsilon, t_{\upsilon+1}]$ when the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1}$ and the generation υ player is of type ω_υ, the type $\omega_{\upsilon-1}$ generation $\upsilon - 1$ player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_\upsilon}^{t_{\upsilon+1}} g^{\upsilon-1(\omega_{\upsilon-1})}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]e^{-r(s-t_\upsilon)}ds \right.$$
$$\left. +e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon-1(\omega_{\upsilon-1})}[t_{\upsilon+1}, x(t_{\upsilon+1})] \,\middle|\, x(t_\upsilon) = x \in X \right\}, \tag{2.5}$$

and the type ω_υ generation υ player seeks to maximize the expected payoff:

$$E\left\{ \int_{t_\upsilon}^{t_{\upsilon+1}} g^{\upsilon(\omega_\upsilon)}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]e^{-r(s-t_\upsilon)}ds \right.$$
$$\left. +e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1}, x(t_{\upsilon+1})] \,\middle|\, x(t_\upsilon) = x \in X \right\}, \tag{2.6}$$

subject to the stochastic dynamics

$$\frac{dx(s)}{ds} = f[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)]ds + \sigma[s, x(s)]dz(s), \quad x(t_\upsilon) = x,$$

for $s \in [t_\upsilon, t_{\upsilon+1}]$.

The game formulated in (2.1)-(2.6) is an extension the Yeung (2011) analysis to a game with stochastic dynamics. It has the characteristics of the finite overlapping generations version of Jrgensen and Yeung's (2005) infinite generations game.

2.2. Noncooperative Outcomes

To obtain a characterization of a noncooperative solution to the asynchronous horizons game in Section 2.1 we first consider the solutions of the games in the last time interval $[t_\upsilon, t_{\upsilon+1}]$, that is the game (2.5)-(2.6). One way to characterize and derive a feedback solution to the games in $[t_\upsilon, t_{\upsilon+1}]$ is to invoke the conventional approach in solving a standard stochastic differential game and obtain:

Lemma 2.1. *If the generation $\upsilon - 1$ player is of type $\omega_{\upsilon-1} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_{\upsilon-1}}\}$ and the generation υ player is of type $\omega_\upsilon \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_\upsilon}\}$ in the time interval $[t_\upsilon, t_{\upsilon+1}]$, a set of feedback strategies $\{\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x); \phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)\}$ constitutes a Nash equilibrium solution for the game (5)-(6), if there exist twice continuously differentiable functions $V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x) : \ [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ and $V^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x) : \ [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ satisfying the following partial differential equations:*

$$-V_t^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)$$

$$= \max_{u_{\upsilon-1}} \Big\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)]e^{-r(t-t_\upsilon)}$$

$$+V_x^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)\ f[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)] \Big\},$$

$$V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x),\ \textit{and}$$

$$-V_t^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)$$

$$= \max_{u_\upsilon} \Big\{ g^{\upsilon(\omega_\upsilon)}[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon]e^{-r(t-t_\upsilon)}$$

$$+V_x^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)\ f[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon] \Big\},$$

$$V^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1},x(t_{\upsilon+1})]. \tag{2.7}$$

Proof. Follow the proof of Theorem 6.27 in Chapter 6 of Basar and Olsder (1999). □

For ease of exposition and sidestepping the issue of multiple equilibria, the analysis focuses on solvable games in which a particular noncooperative Nash equilibrium is chosen by the players in the entire subgame.

We proceed to examine the game in the second last interval $[t_{\upsilon-1},t_\upsilon)$. If the generation $\upsilon-2$ player is of type $\omega_{\upsilon-2}\in\{\omega_1,\omega_2,\cdots,\omega_\varsigma\}$ and the generation $\upsilon-1$ player is of type $\omega_{\upsilon-1}\in\{\omega_1,\omega_2,\cdots,\omega_\varsigma\}$. The type $\omega_{\upsilon-2}$ generation $\upsilon-2$ player seeks to maximize:

$$E\Big\{ \int_{t_{\upsilon-1}}^{t_\upsilon} g^{\upsilon-2(\omega_{\upsilon-2})}[s,x(s),u_{\upsilon-2}^{(\omega_{\upsilon-2},O)\omega_{\upsilon-1}}(s),u_{\upsilon-1}^{(\omega_{\upsilon-1},Y)\omega_{\upsilon-2}}(s)]e^{-r(s-t_{\upsilon-1})}ds$$

$$+e^{-r(t_\upsilon-t_{\upsilon-1})}q^{\upsilon-2(\omega_{\upsilon-2})}[t_\upsilon,x(t_\upsilon)] \Big| x(t_{\upsilon-1})=x\in X \Big\}. \tag{2.8}$$

As shown in Jørgensen and Yeung (2005) the terminal condition of the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player in the game interval $[t_{\upsilon-1},t_\upsilon)$ can be expressed as:

$$\sum_{\alpha=1}^{\zeta_\upsilon} \lambda_\alpha V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\alpha}(t_\upsilon,x). \tag{2.9}$$

Therefore the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player then seeks to maximize:

$$E\left\{ \int_{t_{\upsilon-1}}^{t_\upsilon} g^{\upsilon-1(\omega_{\upsilon-1})}[s,x(s),u_{\upsilon-2}^{(\omega_{\upsilon-2},O)\omega_{\upsilon-1}}(s),u_{\upsilon-1}^{(\omega_{\upsilon-1},Y)\omega_{\upsilon-2}}(s)]e^{-r(s-t_{\upsilon-1})}ds \right.$$

$$\left. +e^{-r(t_\upsilon-t_{\upsilon-1})}\sum_{\alpha=1}^{\zeta_\upsilon}\lambda_\alpha V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\alpha}(t_\upsilon,x(t_\upsilon))\,\middle|\, x(t_{\upsilon-1})=x\in X \right\}.$$

Similarly, the terminal condition of the type ω_k generation k player in the game interval $[t_k,t_{k+1})$ can be expressed as:

$$\sum_{\alpha=1}^{\varsigma_{k+1}}\lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x),\quad \text{for}\quad k\in\{1,2,\cdots,\upsilon-3\}. \tag{2.10}$$

Consider the game in the time interval $[t_k,t_{k+1})$ involving the type ω_k generation k player and the type ω_{k-1} generation $k-1$ player, for $k\in\{1,2,\cdots,\upsilon-3\}$. The type ω_{k-1} generation $k-1$ player will maximize the payoff

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k-1(\omega_{k-1}]}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\varpi_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]\;e^{-r(s-t_k)}ds \right.$$

$$\left. +e^{-r(t_{k+1}-t_k)}q^{k-1(\omega_{k-1})}[t_{k+1},x(t_{k+1})]\,\middle|\, x(t_k)=x\in X \right\}, \tag{2.11}$$

and the type ω_k generation k player will maximize the expected payoff:

$$E\left\{ \int_{t_k}^{t_{k+1}} g^{k(\omega_k)}[s,x(s),u_{k-1}^{(\omega_{k-1},O)\omega_k}(s),u_k^{(\omega_k,Y)\omega_{k-1}}(s)]\;e^{-r(s-t_k)}ds \right.$$

$$\left. +e^{-r(t_{k+1}-t_k)}\sum_{\alpha=1}^{\varsigma_{k+1}}\lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x)\,\middle|\, x(t_k)=x\in X \right\} \tag{2.12}$$

subject to (2.2) with $x(t_k)=x$.

A Nash equilibrium solution to the game (2.11)-(2.12) can be characterized as:

Lemma 2.2. *A set of feedback strategies* $\{\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x);\quad \phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)\}$ *constitutes a Nash equilibrium solution for the game (2.11)-(2.12), if there exist continuously differentiable functions* $V^{k-1(\omega_{k-1},O)\omega_k}(t,x):\quad [t_k,t_{k+1}]\times R^m\to R$ *and* $V^{k(\omega_k,Y)\omega_{k-1}}(t,x):\quad [t_k,t_{k+1}]\times R^m\to R$ *satisfying the following partial differential equations:*

$$-V_t^{k-1(\omega_{k-1},O)\omega_k}(t,x)-\frac{1}{2}\sum_{h,\zeta=1}^{m}\Omega^{h\zeta}(t,x)V_{x^hx^\zeta}^{k-1(\omega_{k-1},O)\omega_k}(t,x)$$

$$=\max_{u_{k-1}}\Big\{\;g^{k-1(\omega_{k-1})}[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)]e^{-r(t-t_k)}$$

$$+V_x^{k-1(\omega_{k-1},O)\omega_k}\; f[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)] \Big\},$$

$$V^{k-1(\omega_{k-1},O)\omega_k}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} q^{k-1(\omega_{k-1})}(t_{k+1},x), \quad \textit{and}$$

$$-V_t^{k(\omega_k,Y)\omega_{k-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{k(\omega_k,Y)\omega_{k-1}}(t,x)$$

$$= \max_{u_k} \Big\{ g^{(k,\omega_k)}[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k] e^{-r(t-t_k)}$$

$$+V_x^{k(\omega_k,Y)\omega_{k-1}}\; f[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k] \Big\},$$

$$V^{k(\omega_k,Y)\omega_{k-1}}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)} \sum_{\alpha=1}^{\varsigma_{k+1}} \lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x),$$

$$\textit{for} \quad k \in \{1,2,\cdots,\upsilon-1\}. \tag{2.13}$$

Proof. Again follow the proof of Theorem 6.16 in Chapter 6 of Basar and Olsder (1999). □

A theorem characterizing the noncooperative outcomes of the game (2.2)-(2.6) can be obtained as:

Theorem 2.1. *A set of feedback strategies* $\{\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x);\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)\}$ *constitutes a Nash equilibrium solution for the game (2.2)-(2.6), if there exist continuously differentiable functions* $V^{k-1(\omega_{k-1},O)\omega_k}(t,x) :[t_k,t_{k+1}]\times R^m \to R$ *and* $V^{k(\omega_k,Y)\omega_{k-1}}(t,x) :[t_k,t_{k+1}]\times R^m \to R$ *satisfying the following partial differential equations:*

$$-V_t^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)$$

$$= \max_{u_{\upsilon-1}} \Big\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)] e^{-r(t-t_\upsilon)}$$

$$+V_x^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x)\; f[t,x,u_{\upsilon-1},\phi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)] \Big\},$$

$$V^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x), \quad \textit{and}$$

$$-V_t^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) V_{x^h x^\zeta}^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)$$

$$= \max_{u_\upsilon} \Big\{ g^{\upsilon(\omega_\upsilon)}[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon] e^{-r(t-t_\upsilon)}$$

$$+V_x^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)\; f[t,x,\phi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x),u_\upsilon] \Big\},$$

$$V^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)}q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1},x(t_{\upsilon+1})]; \tag{2.14}$$

$$-V_t^{k-1(\omega_{k-1},O)\omega_k}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)V_{x^h x^\zeta}^{k-1(\omega_{k-1},O)\omega_k}(t,x)$$

$$= \max_{u_{k-1}}\Big\{ g^{k-1(\omega_{k-1})}[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)]e^{-r(t-t_k)}$$

$$+V_x^{k-1(\omega_{k-1},O)\omega_k}\ f[t,x,u_{k-1},\phi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)]\Big\},$$

$$V^{k-1(\omega_{k-1},O)\omega_k}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)}q^{k-1(\omega_{k-1})}(t_{k+1},x), \textit{ and}$$

$$-V_t^{k(\omega_k,Y)\omega_{k-1}}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x)V_{x^h x^\zeta}^{k(\omega_k,Y)\omega_{k-1}}(t,x)$$

$$= \max_{u_k}\Big\{ g^{(k,\omega_k)}[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k]e^{-r(t-t_k)}$$

$$+V_x^{k(\omega_k,Y)\omega_{k-1}}\ f[t,x,\phi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x),u_k]\Big\},$$

$$V^{k(\omega_k,Y)\omega_{k-1}}(t_{k+1},x) = e^{-r(t_{k+1}-t_k)}\sum_{\alpha=1}^{\varsigma_{k+1}}\lambda_\alpha V^{k(\omega_k,O,)\omega_\alpha}(t_{k+1},x),$$

$$\textit{for} \quad k\in\{1,2,\cdots,\upsilon-1\}. \tag{2.15}$$

Proof. The results (2.14) follows from Lemma 1 and those in (2.15) follows from Lemma 2.2. □

Using Theorem 2.1 one can obtain a non-cooperative game equilibrium of the game (2.2)-(2.6).

3. Dynamic Cooperation among Coexisting Players

Now consider the case when coexisting players want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. The agreement on how to act cooperatively and allocate cooperative payoff constitutes the solution optimality principle of a cooperative scheme. In particular, the solution optimality principle for the cooperative game includes (i) an agreement on a set of cooperative strategies/controls, and (ii) an imputation of their payoffs.

Consider the game in the time interval $[t_k,t_{k+1})$ involving the type ω_k generation k player and the type ω_{k-1} generation $k-1$ player. Let $\varpi_\ell^{(\omega_{k-1},\omega_k)}$ denote the probability that the type ω_k generation k player and the type ω_{k-1} generation $k-1$ player would agree to the solution imputation

$[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x),\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x)]$ over the time interval $[t_k,t_{k+1})$,

where $\sum_{h=1}^{\varsigma(\omega_{k-1},\omega_k)} \varpi_\ell^{(\omega_{k-1},\omega_k)} = 1$.

At time t_1, the agreed-upon imputation for the type ω_1 generation 0 player and the type ω_2 generation 1 player are known.

The solution imputation may be governed by many specific principles. For instance, the players may agree to maximize the sum of their expected payoffs and equally divide the excess of the cooperative payoff over the noncooperative payoff. As another example, the solution imputation may be an allocation principle in which the players allocate the total joint payoff according to the relative sizes of the players' noncooperative payoffs. Finally, it is also possible that the players refuse to cooperate. In that case, the imputation vector becomes $[V^{k-1(\omega_{k-1},O)\omega_k}(t,x), V^{k(\omega_k,Y)\omega_{k-1}}(t,x)]$.

Both group optimality and individual rationality are required in a cooperative plan. Group optimality requires the players to seek a set of cooperative strategies/controls that yields a Pareto optimal solution. The allocation principle has to satisfy individual rationality in the sense that neither player would be no worse off than before under cooperation.

3.1. Group Optimality

Since payoffs are transferable, group optimality requires the players coexisting in the same time interval to maximize their expected joint payoff. Consider the last time interval $[t_\upsilon, t_{\upsilon+1}]$, in which the generation $\upsilon-1$ player is of type $\omega_{\upsilon-1} \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$ and the generation υ player is of type $\omega_\upsilon \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$. The players maximize their expected joint payoff:

$$E\Bigg\{ \int_{t_\upsilon}^{t_{\upsilon+1}} \Big(g^{\upsilon-1(\omega_{\upsilon-1})}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)] + g^{\upsilon(\omega_\upsilon)}[s, x(s), u_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(s), u_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(s)] \Big) e^{-r(s-t_\upsilon)} ds$$
$$+ e^{-r(t_{\upsilon+1}-t_\upsilon)} \Big(q^{\upsilon-1(\omega_{\upsilon-1})}[t_{\upsilon+1}, x(t_{\upsilon+1})] + q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1}, x(t_{\upsilon+1})] \Big) \Bigg| \, x(t_\upsilon) = x \in X \Bigg\}, \tag{3.1}$$

subject to (2.2) with $x(t_\upsilon) = x$.

Invoking the technique of stochastic dynamic programming an optimal solution of the problem (3.1)-(2.2) can be characterized as:

Lemma 3.1. *A set of Controls $\{\psi_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon}(t,x); \psi_\upsilon^{(\omega_\upsilon,Y)\omega_{\upsilon-1}}(t,x)\}$ constitutes an optimal solution for the stochastic control problem (3.1)-(2.2), if there exist continuously differentiable functions $W^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) : [t_\upsilon, t_{\upsilon+1}] \times R^m \to R$ satisfying the following partial differential equations:*

$$
\begin{aligned}
&-W_t^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) W_{x^h x^\zeta}^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) \\
&\qquad = \max_{u_{\upsilon-1},u_\upsilon} \Bigg\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t,x,u_{\upsilon-1},u_\upsilon]\, e^{-r(t-t_\upsilon)} \\
&+ g^{\upsilon(\omega_\upsilon)}[t,x,u_{\upsilon-1},u_\upsilon] e^{-r(t-t_\upsilon)} + W_x^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) f[t,x,u_{\upsilon-1},u_\upsilon] \Bigg\}, \\
&W^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t_{\upsilon+1},x) = e^{-r(t_{\upsilon+1}-t_\upsilon)}[q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1},x) + q^{\upsilon(\omega_\upsilon)}(t_{\upsilon+1},x)].
\end{aligned} \tag{3.2}
$$

Proof. The results in (3.2) are the characterization of optimal solution to the stochastic control problem (3.1)-(2.2) according to stochastic dynamic programming. □

We proceed to examine joint payoff maximization problem in the time interval $[t_{\upsilon-1}, t_\upsilon)$ involving the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player and type $\omega_{\upsilon-2}$ generation $\upsilon-2$ player. A critical problem is to determent the expected terminal valuation to the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player at time t_υ in the optimization problem within the time interval $[t_{\upsilon-1}, t_\upsilon)$. By time t_υ, the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player may co-exist with the $\omega_\upsilon \in \{\omega_1, \omega_2, \cdots, \omega_\varsigma\}$ generation υ player with probabilities $\{\lambda_1, \lambda_2, \cdots, \lambda_\varsigma\}$. Consider the case in the time interval $[t_\upsilon, t_{\upsilon+1})$ in which the type $\omega_{\upsilon-1}$ generation $\upsilon-1$ player and the type ω_υ generation υ player co-exist. The probability that the type $\omega_{\upsilon-1}$ generation player and the type ω_υ generation player would agree to the solution imputation

$$
[\xi^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon[h]}(t,x), \xi^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}[h]}(t,x)] \quad \text{is} \quad \varpi_h^{(\omega_{\upsilon-1},\omega_\upsilon)} \quad \text{where} \quad \sum_h \varpi_h^{(\omega_{\upsilon-1},\omega_\upsilon)} = 1. \tag{3.3}
$$

In the optimization problem within the time interval $[t_{\upsilon-1}, t_\upsilon)$, the expected terminal reward to the $\omega_{\upsilon-1}$ generation $\upsilon-1$ player at time t_υ can be expressed as:

$$
\sum_{\alpha=1}^{\varsigma_\upsilon} \sum_{h=1}^{\varsigma(\omega_{\upsilon-1},\omega_\alpha)} \varpi_h^{(\omega_{\upsilon-1},\omega_\alpha)} \xi^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\alpha[h]}(t_\upsilon,x). \tag{3.4}
$$

Similarly for the optimization problem within the time interval $[t_k, t_{k+1})$, the expected terminal reward to the ω_k generation k player at time t_{k+1} can be expressed as:

$$
\sum_{\alpha=1}^{\varsigma_{k+1}} \sum_{h=1}^{\varsigma(\omega_k,\omega_\alpha)} \varpi_h^{(\omega_k,\omega_\alpha)} \xi^{k(\omega_k,O)\omega_\alpha[h]}(t_{k+1},x), \quad \text{for} \quad k \in \{1,2,\cdots,H-3\}. \tag{3.5}
$$

The joint maximization problem in the time interval $[t_k, t_{k+1})$, for $k \in \{1,2,\cdots,\upsilon-3\}$, involving the type ω_k generation k player and type ω_{k-1} generation $k-1$ player can be expressed as:

$$
\begin{aligned}
&\max_{u_{k-1},u_k} E\Bigg\{ \int_{t_k}^{t_{k+1}} \Bigg(g^{k-1(\omega_{k-1})}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \\
&\qquad + g^{k(\omega_k)}[s, x(s), u_{k-1}^{(\omega_{k-1},O)\omega_k}(s), u_k^{(\omega_k,Y)\omega_{k-1}}(s)] \Bigg) e^{-r(s-t_k)} ds \\
&\qquad\qquad + e^{-r(t_{k+1}-t_k)} \Bigg(q^{k-1(\omega_{k-1})}[t_{k+1}, x(t_{k+1})] \\
&+ \sum_{\alpha=1}^{\varsigma_{k+1}} \sum_{h=1}^{\varsigma(\omega_k,\omega_\alpha)} \varpi_h^{(\omega_k,\omega_\alpha)}\ \xi^{k(\omega_k,O)\omega_\alpha[h]}(t_{k+1}, x(t_{k+1})) \Bigg) \Bigg|\, x(t_k) = x \in X \Bigg\},
\end{aligned} \tag{3.6}
$$

subject to (2.2) with $x(t_k) = x$.

The conditions characterizing an optimal solution of the problem (3.6)-(2.2) are given as follows.

Theorem 3.1. *A set of Controls $\{\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x);\psi_k^{(\omega_k,Y)\omega_{k-1}}(t,x)\}$ constitutes an optimal solution for the stochastic control problem (3.6)-(2.2), if there exist continuously differentiable functions $W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x) : [t_k, t_{k+1}) \times R^m \to R$ satisfying the following partial differential equations:*

$$
-W_t^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) W_{x^h x^\zeta}^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x)
$$

$$
= \max_{u_{\upsilon-1},u_\upsilon} \Bigg\{ g^{\upsilon-1(\omega_{\upsilon-1})}[t, x, u_{\upsilon-1}, u_\upsilon]\, e^{-r(t-t_\upsilon)}
$$

$$
+ g^{\upsilon(\omega_\upsilon)}[t, x, u_{\upsilon-1}, u_\upsilon] e^{-r(t-t_\upsilon)} + W_x^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t,x) f[t, x, u_{\upsilon-1}, u_\upsilon] \Bigg\},
$$

$$
W^{[t_\upsilon,t_{\upsilon+1}](\omega_{\upsilon-1},\omega_\upsilon)}(t_{\upsilon+1}, x) = e^{-r(t_{\upsilon+1}-t_\upsilon)}[q^{\upsilon-1(\omega_{\upsilon-1})}(t_{\upsilon+1}, x) + q^{\upsilon(\omega_\upsilon)}(t_{\upsilon+1}, x)];
$$

for $k \in \{1, 2, \cdots, \upsilon - 1\}$:

$$
-W_t^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x) W_{x^h x^\zeta}^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x)
$$

$$
= \max_{u_{k-1},u_k} \{ g^{k-1(\omega_{k-1})}[t, x, u_{k-1}, u_k] e^{-r(t-t_k)}
$$

$$
+ g^{k(\omega_k)}[t, x, u_{k-1}, u_k] e^{-r(t-t_k)} + W_x^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t,x)\ f[t, x, u_{k-1}, u_k] \},
$$

$$
\begin{aligned}
W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t_{k+1}, x) &= e^{-r(t_{k+1}-t_k)} \Bigg(q^{k-1(\omega_{k-1})}(t_{k+1}, x) \\
&+ \sum_{\alpha=1}^{\varsigma_{k+1}} \sum_{h=1}^{\varsigma(\omega_k,\omega_\alpha)} \varpi_h^{(\omega_k,\omega_\alpha)}\ \xi^{k(\omega_k,O)\omega_\alpha[h]}(t_{k+1}, x) \Bigg).
\end{aligned} \tag{3.7}
$$

Proof. Invoking the standard technique of stochastic dynamic programming we obtain the conditions characterizing an optimal solution of the problem (3.6)-(2.2) as in (3.7). $\square$

Substituting the set of cooperative strategies into (2.2) yields the dynamics of the cooperative state trajectory in the time interval $[t_k, t_{k+1})$

$$\frac{dx(s)}{ds} = f[s, x(s), \psi_{k-1}^{(\omega_{k-1},O)\omega_k}(s, x(s)), \psi_k^{(\omega_k,Y)\omega_{k-1}}(s, x(s))] + \sigma[s, x(s)]dz(s), \quad (3.8)$$

for $s \in [t_k, t_{k+1})$, $k \in \{1, 2, \cdots, \upsilon\}$ and $x(t_1) = x_0$.

We denote the set of realizable states at time t from (3.8) under the scenarios of different players by $X_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$, for $t \in [t_k, t_{k+1})$ and $k \in \{1, 2, \cdots, \upsilon\}$. We use the term $x_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*} \in X_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$ to denote an element in $X_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$. The term x_t^* is used to denote $x_t^{\{t_k,t_{k+1}](\omega_k,\omega_{k+1})*}$ whenever there is no ambiguity

To fulfill group optimality, the imputation vectors have to satisfy:

$$\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t, x_t^*) + \xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t, x_t^*) = W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t, x_t^*), \quad (3.9)$$

for $t \in [t_k, t_{k+1})$, $\omega_k \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$, $\omega_{k-1} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_{k-1}}\}$, $\ell \in \{1, 2, \cdots, \varsigma_{(\omega_{k-1},\omega_k)}\}$ and $k \in \{0, 1, 2, \cdots, \upsilon\}$,

where x_t^* is the short form for $x_t^{(\omega_{k-1},\omega_k)*}$.

3.2. Individual Rationality

In a dynamic framework, individual rationality requires that the imputation received by a player has to be no less than his noncooperative payoff throughout the time interval in concern. Hence for individual rationality to hold along the cooperative trajectory $\left\{ x^{(\omega_{k-1},\omega_k)*}(t) \right\}_{t=t_k}^{t_{k+1}}$,

$$\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t, x_t^*) \geq V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) \quad \text{and}$$

$$\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t, x_t^*) \geq V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*), \quad (3.10)$$

for $t \in [t_k, t_{k+1})$, $\omega_k \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_k}\}$, $\omega_{k-1} \in \{\omega_1, \omega_2, \cdots, \omega_{\varsigma_{k-1}}\}$, $\ell \in \{1, 2, \cdots, \varsigma_{(\omega_{k-1},\omega_k)}\}$ and $k \in \{0, 1, 2, \cdots, \upsilon\}$.

For instance, an imputation vector equally dividing the excess of the cooperative payoff over the noncooperative payoff can be expressed as:

$$\begin{aligned}\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t, x_t^*) &= V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) + 0.5[W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t, x_t^*) \\ &\quad - V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) - V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*)], \quad \text{and} \\ \xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t, x_t^*) &= V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*) + 0.5[W^{[t_k,t_{k+1}](\omega_{k-1},\omega_k)}(t, x_t^*) \\ &\quad - V^{k-1(\omega_{k-1},O)\omega_k}(t, x_t^*) - V^{k(\omega_k,Y)\omega_{k-1}}(t, x_t^*)]. \quad (3.11)\end{aligned}$$

One can readily see that the imputations in (3.11) satisfy individual rationality and group optimality.

4. Subgame Consistent Solutions and Payoff Distribution

A stringent requirement for solutions of cooperative stochastic differential games to be dynamically stable is the property of subgame consistency. Under subgame consistency, an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. In particular, when the game proceeds, at each instant of time the players are guided by the same optimality principles, and hence do not have any ground for deviation from the previously adopted optimal behavior throughout the game.

According to the solution optimality principle the players agree to share their cooperative payoff according to the imputations

$$[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*),\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*)] \tag{4.1}$$

over the time interval $[t_k, t_{k+1})$.

To achieve dynamic consistency, a payment scheme has to be derived so that imputation (4.1) will be maintained throughout the time interval $[t_k, t_{k+1})$. Following Yeung and Petrosyan (2004 and 2006) and Yeung (2011), we formulate a payoff distribution procedure (PDP) over time so that the agreed imputations (4.1) can be realized. Let $B_{k-1}^{(\omega_{k-1},O)\omega_k[\ell]}(s)$ and $B_k^{(\omega_k,Y)\omega_{k-1}[\ell]}(s)$ denote the instantaneous payments at time $s \in [t_k, t_{k+1})$ allocated to the type ω_{k-1} generation $k-1$ (old) player and type ω_k generation k (young) player.

In particular, the imputation vector can be expressed as:

$$\begin{aligned}
\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*) &= E\left\{ \int_{t_k}^{t_{k+1}} B_{k-1}^{(\omega_{k-1},O)\omega_k[\ell]}(s)\, e^{-r(s-t_k)}ds \right.\\
&\quad \left. + e^{-r(t_{k+1}-t_k)} q^{k-1(\omega_{k-1})}[t_{k+1},x^*(t_{k+1})] \,\middle|\, x(t_k)=x_t^* \in X \right\},\\
\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) &= E\left\{ \int_{t_k}^{t_{k+1}} B_{k}^{(\omega_k,Y)\omega_{k-1}[\ell]}(s)\, e^{-r(s-t_k)}ds \right.\\
&\quad \left. + \sum_{\alpha=1}^{\varsigma} \sum_{\ell=1}^{\varsigma_{(\omega_k,\omega_\alpha)}} \varpi_\ell^{(\omega_k,\omega_\alpha)} \xi^{k(\omega_k,O)\omega_\alpha[\ell]}(t_{k+1},x^*(t_{k+1})) \,\middle|\, x(t_k)=x_t^* \in X \right\},
\end{aligned} \tag{4.2}$$

for $k \in \{1,2,\cdots,\upsilon-1\}$, and

$$\begin{aligned}
\xi^{\upsilon-1(\omega_{\upsilon-1},O)\omega_\upsilon[\ell]}(t,x_t^*) &= E\left\{ \int_{t_\upsilon}^{t_{\upsilon+1}} B_{\upsilon-1}^{(\omega_{\upsilon-1},O)\omega_\upsilon[\ell]}(s)\, e^{-r(s-t_\upsilon)}ds \right.\\
&\quad \left. + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon-1(\omega_{\upsilon-1})}[t_{\upsilon+1},x^*(t_{\upsilon+1})] \,\middle|\, x(t_\upsilon)=x_t^* \in X \right\},\\
\xi^{\upsilon(\omega_\upsilon,Y)\omega_{\upsilon-1}[\ell]}(t,x_t^*) &= E\left\{ \int_{t_\upsilon}^{t_{\upsilon+1}} B_{\upsilon}^{(\omega_\upsilon,Y)\omega_{\upsilon-1}[\ell]}(s)\, e^{-r(s-t_\upsilon)}ds \right.\\
&\quad \left. + e^{-r(t_{\upsilon+1}-t_\upsilon)} q^{\upsilon(\omega_\upsilon)}[t_{\upsilon+1},x^*(t_{\upsilon+1})] \,\middle|\, x(t_\upsilon)=x_t^* \in X \right\}.
\end{aligned} \tag{4.3}$$

Using the analysis in Yeung and Petrosyan (2006) and Petrosyan and Yeung (2007) we obtain:

Theorem 4.1. *If the imputation vector* $[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*),\xi^{k(\omega_k,O)\omega_{k-1}[\ell]}(t,x_t^*)]$ *are functions that are continuously differentiable in* t *and* x_t^**, a PDP with an instantaneous payment at time* $t \in [t_k, t_{k+1})$*:*

$$\begin{aligned}
B_{k-1}^{(\omega_{k-1},O)\omega_k[\ell]}(t) = -\xi_t^{k-1(\omega_{k-1},O)\omega_k[\wp]}(t,x_t^*) \\
- \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x_t^*)\xi_{x^h x^\zeta}^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*) \\
- \xi_x^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*) f[t,x_t^*,\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x_t^*),\psi_k^{(\omega_k,Y)\omega_{k-1}}(t,x_t^*)] \quad (4.4)
\end{aligned}$$

allocated to the type ω_{k-1} *generation* $k-1$ *player;*
and an instantaneous payment at time $t \in [t_k, t_{k+1})$*:*

$$\begin{aligned}
B_k^{(\omega_k,Y)\omega_{k-1}[\ell]}(t) = -\xi_t^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x_t^*)\xi_{x^h x^\zeta}^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) \\
-\xi_x^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*) f[t,x_t^*,\psi_{k-1}^{(\omega_{k-1},O)\omega_k}(t,x_t^*),\psi_k^{(\omega_k,Y)\omega_{k-1}}(t,x_t^*)]
\end{aligned}$$

allocated to the type ω_k *generation* k *player,*
yields a mechanism leading to the realization of the imputation vector

$$[\xi^{k-1(\omega_{k-1},O)\omega_k[\ell]}(t,x_t^*),\xi^{k(\omega_k,Y)\omega_{k-1}[\ell]}(t,x_t^*)],$$

for $\ell \in \{1,2,\cdots,\varsigma_{(\omega_{k-1},\omega_k)}\}$ *and* $k \in \{1,2,\cdots,\upsilon\}$.

Proof. Follow the proof leading to Theorem 4.4.1 in Yeung and Petrosyan (2006) with the imputation vector in present value (rather than in current value). ☐

5. An Illustration in Resource Extraction

Consider the game in which there are 4 overlapping generations of players with generation 0 and generation 1 players in $[t_1, t_2)$, generation 1 and generation 2 players in $[t_2, t_3)$, generation 2 and generation 3 players in $[t_3, t_4]$. Players are of either type 1 or type 2. The instantaneous payoffs and terminal rewards of the type 1 generation k player and the type 2 generation k player are respectively:

$$\left[(u_k)^{1/2} - \frac{c_1}{x^{1/2}}u_k\right] \quad \text{and} \quad q_1 x^{1/2}; \quad \text{and} \quad \left[(u_k)^{1/2} - \frac{c_2}{x^{1/2}}u_k\right] \quad \text{and} \quad q_2 x^{1/2}. \tag{5.1}$$

At initial time t_1, it is known that the generation 0 player is of type 1 and the generation 1 player is of type 2. It is also known that the generation 2 and generation 3 players may be of type 1 with probability $\lambda_1 = 0.4$ and of type 2 with probability $\lambda_2 = 0.6$.

The state dynamics of the game is characterized by the stochastic dynamics:

$$\frac{dx(s)}{ds} = [ax(s)^{1/2} - bx(s) - u_{k-1}(s) - u_k(s)]ds + \sigma x(s)dz(s),\ x(t_1) = x_0 \in X \subset R, \tag{5.2}$$

for $s \in [t_k, t_{k+1})$ and $k \in \{1, 2, 3\}$.

The game is an asynchronous horizons version of the synchronous-horizon resource extraction game in Yeung and Petrosyan (2006) and an extension of the Yeung (2011) analysis to include stochastic dynamics. The state variable $x(s)$ is the biomass of a renewable resource. $u_k(s)$ is the harvest rate of the generation k extraction firm. The death rate of the resource is b. The rate of growth is $a/x^{1/2}$ which reflects the decline in the growth rate as the biomass increases. The type $i \in \{1, 2\}$ generation k extraction firm's extraction cost is $c_i u_k(s)x(s)^{-1/2}$.

This asynchronous horizon game can be expressed as follows. In the time interval $[t_k, t_{k+1})$, for $k \in \{1, 2\}$, consider the case with a type $i \in \{1, 2\}$ generation $k-1$ firm and a type $j \in \{1, 2\}$ generation k firm, the game becomes

$$
\begin{aligned}
&\max_{u_{k-1}} E\left\{ \int_{t_k}^{t_{k+1}} \left[[u_{k-1}^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_1^{(i,O)j}(s) \right] \exp[-r(s-t_k)]\, ds \right.\\
&\qquad\qquad \left. + \exp\left[-r(t_{k+1}-t_k)\right] q_i x(t_{k+1})^{\frac{1}{2}} \right\},\\
&\max_{u_k} E\left\{ \int_{t_k}^{t_{k+1}} \left[[u_k^{(j,Y)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,Y)i}(s) \right] \exp[-r(s-t_k)]\, ds \right.\\
&\quad + \sum_{\alpha=1}^{2} \lambda_\alpha \int_{t_3}^{t_4} \left[[u_k^{(j,O)\alpha}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_k^{(j,O)\alpha}(s) \right] \exp[-r(s-t_k)]\, ds\\
&\qquad\qquad \left. + \exp\left[-r(t_{k+2}-t_k)\right] q_j x(t_{k+2})^{\frac{1}{2}} \right\},
\end{aligned} \tag{5.3}
$$

subject to (5.2).

In the time interval $[t_3, t_4]$, in the case with a type $i \in \{1, 2\}$ generation 2 firm and a type $j \in \{1, 2\}$ generation 3 firm, the game becomes

$$
\begin{aligned}
&\max_{u_2} E\left\{ \int_{t_3}^{t_4} \left[[u_2^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_2^{(i,O)j}(s) \right] \exp[-r(s-t_3)]\, ds \right.\\
&\qquad\qquad \left. + \exp\left[-r(t_4-t_3)\right] q_i x(t_4)^{\frac{1}{2}} \,\middle|\, x(t_3) = x \right\},\\
&\max_{u_3} E\left\{ \int_{t_3}^{t_4} \left[[u_3^{(j,O)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,O)i}(s) \right] \exp[-r(s-t_3)]\, ds \right.\\
&\qquad\qquad \left. + \exp\left[-r(t_4-t_3)\right] q_j x(t_4)^{\frac{1}{2}} \,\middle|\, x(t_3) = x \right\},
\end{aligned} \tag{5.4}
$$

subject to (5.2) with $x(t_3) = x$.

5.1. Noncooperative Outcomes

In this section we characterize the noncooperative outcome of the asynchronous horizons game (5.2)-(5.4).

Proposition 5.1. *The value functions for the type $i \in \{1, 2\}$ generation $k-1$ firm and the type $j \in \{1, 2\}$ generation k firm coexisting in the game interval $[t_k, t_{k+1})$ can be obtained as:*

$$V^{k-1(i,O)j}(t,x) = \exp[-r(t-t_k)] \left[A_{k-1}^{(i,O)j}(t)x^{1/2} + C_{k-1}^{(i,O)j}(t)\right], \text{ and}$$

$$V^{k(j,Y)i}(t,x) = \exp[-r(t-t_k)] \left[A_k^{(j,Y)i}(t)x^{1/2} + C_k^{(j,Y)i}(t)\right], \tag{5.5}$$

for $k \in \{1,2,3\}$ *and* $i,j \in \{1,2\}$,
where
$A_{k-1}^{(i,O)j}(t)$, $C_{k-1}^{(i,O)j}(t)$, $A_k^{(j,Y)i}(t)$ *and* $C_k^{(j,Y)i}(t)$ *satisfy:*

$$\dot{A}_{k-1}^{(i,O)j}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8}\right] A_{k-1}^{(i,O)j}(t) - \frac{1}{2\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]} + \frac{c_i}{4\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2}$$

$$+ \frac{A_{k-1}^{(i,O)j}(t)}{8\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2} + \frac{A_{k-1}^{(i,O)j}(t)}{8\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2},$$

$$\dot{C}_{k-1}^{(i,O)j}(t) = rC_{k-1}^{(i,O)j}(t) - \frac{a}{2}A_{k-1}^{(i,O)j}(t),$$

$$A_{k-1}^{(i,O)j}(t_{k+1}) = q_i \quad \text{and} \quad C_{k-1}^{(i,O)j}(t_{k+1}) = 0, \quad \text{for} \quad k \in \{1,2,3\}; \tag{5.6}$$

$$\dot{A}_k^{(j,Y)i}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8}\right] A_k^{(j,Y)i}(t) - \frac{1}{2\left[c_j + A_k^{(j,Y)i}(t)/2\right]} + \frac{c_j}{4\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2}$$

$$+ \frac{A_k^{(j,Y)i}(t)}{8\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2} + \frac{A_k^{(j,Y)i}(t)}{8\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2},$$

$$\dot{C}_k^{(j,Y)i}(t) = rC_k^{(j,Y)i}(t) - \frac{a}{2}A_k^{(j,Y)i}(t), \quad \text{for} \quad k \in \{1,2,3\};$$

$$A_k^{(j,Y)i}(t_{k+1}) = e^{-r(t_{k+1}-t_k)} \sum_{\ell=1}^{2} \lambda_\ell A_k^{(j,O)\ell}(t_{k+1}) \quad \text{and}$$

$$C_k^{(j,Y)i}(t_{k+1}) = e^{-r(t_{k+1}-t_k)} \sum_{\ell=1}^{2} \lambda_\ell C_k^{(j,O)\ell}(t_{k+1}),$$

$$\text{for} \quad k \in \{1,2\}, \quad \text{and} \quad A_3^{(j,Y)i}(t_4) = q_j \quad \text{and} \quad C_3^{(j,Y)i}(t_4) = 0. \tag{5.7}$$

Proof. Using Lemmas 2.1 and 2.2 and the analysis in Proposition 5.1.1 in Yeung and Petrosyan (2006), one can obtain the value functions in (5.5). □

Following Yeung and Petrosyan (2006) the game equilibrium strategies can be expressed as:

$$\phi_{k-1}^{(i,O)j}(t,x) = \frac{x}{4\left[c_i + A_{k-1}^{(i,O)j}(t)/2\right]^2} \quad \text{and} \quad \phi_k^{(j,Y)i}(t,x) = \frac{x}{4\left[c_j + A_k^{(j,Y)i}(t)/2\right]^2}. \tag{5.8}$$

A complete characterization of the noncooperative market outcome is provided by Proposition 1 and (38).

5.2. Dynamic Cooperation

Now consider the case when coexisting firms want to cooperate and agree to act and allocate the cooperative payoff according to a set of agreed upon optimality principles. Let there be three acceptable imputations.

Imputation I: the firms would share the excess gain from cooperation equally with weights $w_{k-1}^1 = w_k^1 = 0.5$.

Imputation II: the generation $k-1$ firm acquires $w_{k-1}^2 = 0.6$ of the excess gain from cooperation and the generation k firm acquires $w_k^2 = 0.4$ of the gain.

Imputation III: the generation $k-1$ firm acquires $w_{k-1}^3 = 0.4$ of the excess gain from cooperation and the generation k firm acquires $w_k^3 = 0.6$ of the gain.

In time interval $[t_k, t_{k+1})$, if both the generation $k-1$ firm and the generation k firm are of type 1, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(1,1)} = 0.8$, $\varpi_2^{(1,1)} = 0.1$ and $\varpi_3^{(1,1)} = 0.1$.

If both the generation $k-1$ firm and the generation k firm are of type 2, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(2,2)} = 0.7$, $\varpi_2^{(2,2)} = 0.15$ and $\varpi_3^{(2,2)} = 0.15$.

If the generation $k-1$ firm is of type 1 and the generation k firm are of type 2, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(1,2)} = 0.15$, $\varpi_2^{(1,2)} = 0.75$ and $\varpi_3^{(1,2)} = 0.1$.

If the generation $k-1$ firm is of type 2 and the generation k firm are of type 1, the probabilities that the firms would agree to Imputations I, II and III are respectively $\varpi_1^{(2,1)} = 0.15$, $\varpi_2^{(2,1)} = 0.1$ and $\varpi_3^{(2,1)} = 0.75$.

At initial time t_1, the type 1 generation 0 firm and the type 2 generation 1 firm are assumed to have agreed to Imputation II.

Since payoffs are transferable, group optimality requires the firms coexisting in the same time interval to maximize their joint payoff. Consider the last time interval $[t_3, t_4]$, in which the generation 2 firm is of type $i \in \{1, 2\}$ and the generation 3 firm is of type $j \in \{1, 2\}$. The firms maximize their expected joint profit

$$E\left\{ \int_{t_3}^{t_4} \left[[u_2^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_2^{(i,O)j}(s) \right] \exp[-r(s-t_3)]\, ds \right.$$
$$+ \int_{t_3}^{t_4} \left[[u_3^{(j,O)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_2^{(j,O)i}(s) \right] \exp[-r(s-t_3)]\, ds$$
$$\left. + \exp\left[-r(t_4-t_3)\right] q_i x(t_4)^{\frac{1}{2}} + \exp\left[-r(t_4-t_3)\right] q_j x(t_4)^{\frac{1}{2}} \,\middle|\, x(t_3) = x \right\},$$

subject to (5.2) with $x(t_3) = x$.

Proposition 5.2. *The maximized joint payoff with type $i \in \{1, 2\}$ generation 2 firm and the type $j \in \{1, 2\}$ generation 3 firm coexisting in the game interval $[t_3, t_4)$ can be obtained as:*

$$W^{[t_3,t_4](i,j)}(t,x) = \exp[-r(t-t_3)] \left[A^{[t_3,t_4](i,j)}(t) x^{1/2} + C^{[t_3,t_4](i,j)}(t) \right], \qquad (5.9)$$

where $A^{[t_3,t_4](i,j)}(t)$ and $C^{[t_3,t_4](i,j)}(t)$ satisfy:

$$\dot{A}^{[t_3,t_4](i,j)}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8}\right] A^{[t_3,t_4](i,j)}(t) - \frac{1}{2\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]}$$

$$-\frac{1}{2\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]} + \frac{c_i}{4\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]^2} + \frac{c_j}{4\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]^2}$$

$$+\frac{A^{[t_3,t_4](i,j)}(t)}{8\left[c_i + A^{[t_3,t_4](i,j)}(t)/2\right]^2} + \frac{A^{[t_3,t_4](i,j)}(t)}{8\left[c_j + A^{[t_3,t_4](i,j)}(t)/2\right]^2},$$

$$\dot{C}^{[t_3,t_4](i,j)}(t) = rC^{[t_3,t_4](i,j)}(t) - \frac{a}{2}A^{[t_3,t_4](i,j)}(t),$$

$$A^{[t_3,t_4](i,j)}(t_4) = q_i + q_j \quad \textit{and} \quad C^{[t_3,t_4](i,j)}(t_4) = 0. \tag{5.10}$$

Proof. Using Lemma 3.1 and the analysis in example 5.2.1 in Yeung and Petrosyan (2006), one can obtain (5.9)-(5.10). □

The solution time paths $A^{[t_3,t_4](i,j)}(t)$ and $C^{[t_3,t_4](i,j)}(t)$ for the system of first order differential equations in (39)-(40) can be computed numerically for given values of the model parameters r,q_1, q_2, c_1, c_2, a and b.

In the game interval $[t_3, t_4)$ if type $i \in \{1, 2\}$ generation 2 firm and the type $j \in \{1, 2\}$ generation 3 firm coexisting, the imputations of the firms under cooperation can be expressed as:

$$\xi^{2(i,O)j[\ell]}(t,x) = V^{2(i,O)j}(t,x) + w_2^h[W^{[t_3,t_4](i,j)}(t,x) - V^{2(i,O)j}(t,x) - V^{3(j,Y)i}(t,x)],$$

$$\xi^{3(j,Y)i[\ell]}(t,x) = V^{3(j,Y)i}(t,x) + w_3^h[W^{[t_3,t_4](i,j)}(t,x) - V^{2(i,O)j}(t,x) - V^{3(j,Y)i}(t,x)],$$

$$\text{for} \quad \ell \in \{1, 2, 3\}. \tag{5.11}$$

Now we proceed to the second last interval $[t_k, t_{k+1})$ for $k = 2$. Consider the case in which the generation k firm is of type $j \in \{1, 2\}$ and the generation $k-1$ firm is known to be of type $i = 2$. Following the analysis in (19) and (20), the expected terminal reward to the type j generation k firm at time t_{k+1} can be expressed as:

$$\sum_{\ell=1}^{2} \lambda_\ell \sum_{h=1}^{3} \varpi_h^{(j,\ell)} \xi^{k(j,O)\ell[h]}(t_{k+1}, x), \quad \text{for} \quad k = 2. \tag{5.12}$$

A review of Proposition 5.1, Proposition 5.2 and (5.11) shows the term in (5.12) can be written as:

$$A_k^{\zeta(j,O)} x^{1/2} + C_k^{\zeta(j,O)}, \tag{5.13}$$

where $A_k^{\zeta(j,O)}$ and $C_k^{\zeta(j,O)}$ are constant terms.

The joint maximization problem in the time interval $[t_k, t_{k+1})$, for $k \in \{1, 2\}$, involving the type j generation k player and type i generation $k-1$ player can be expressed as:

$$\max_{u_{k-1}, u_k} E\left\{ \int_{t_k}^{t_{k+1}} \left[[u_{k-1}^{(i,O)j}(s)]^{1/2} - \frac{c_i}{x(s)^{1/2}} u_{k-1}^{(i,O)j}(s) \right] \exp[-r(s-t_k)]\, ds \right.$$

$$+ \int_{t_3}^{t_4} \left[[u_k^{(j,O)i}(s)]^{1/2} - \frac{c_j}{x(s)^{1/2}} u_k^{(j,O)i}(s) \right] \exp[-r(s-t_k)]\, ds$$

$$\left. + \exp\left[-r(t_{k+1}-t_k)\right] \left[q_i x(t_{k+1})^{\frac{1}{2}} + A_k^{\zeta(j,O)} x(t_{k+1})^{1/2} + C_k^{\zeta(j,O)} \right] \right| x(t_k) = x \Bigg\},$$

subject to (5.2).

Proposition 5.3. *The maximized expected joint payoff with type $i \in \{1, 2\}$ generation $k-1$ firm and the type $j \in \{1, 2\}$ generation k firm coexisting in the game interval $[t_k, t_{k+1})$, for $k \in \{1, 2\}$, can be obtained as:*

$$W^{[t_k,t_{k+1}](i,j)}(t, x) = \exp[-r(t-t_k)] \left[A^{[t_k,t_{k+1}](i,j)}(t) x^{1/2} + C^{[t_k,t_{k+1}](i,j)}(t) \right], \tag{5.14}$$

where $A^{[t_k,t_{k+1}](i,j)}(t)$ and $C^{[t_k,t_{k+1}](i,j)}(t)$ satisfy:

$$\dot{A}^{[t_k,t_{k+1}](i,j)}(t) = \left[r + \frac{b}{2} + \frac{\sigma^2}{8} \right] A^{[t_k,t_{k+1}](i,j)}(t) - \frac{1}{2\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]}$$

$$- \frac{1}{2\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]} + \frac{c_i}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} + \frac{c_j}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}$$

$$+ \frac{A^{[t_k,t_{k+1}](i,j)}(t)}{8\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} + \frac{A^{[t_k,t_{k+1}](i,j)}(t)}{8\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2},$$

$$\dot{C}^{[t_k,t_{k+1}](i,j)}(t) = r C^{[t_k,t_{k+1}](i,j)}(t) - \frac{a}{2} A^{[t_k,t_{k+1}](i,j)}(t),$$

$$A^{[t_k,t_{k+1}](i,j)}(t_{k+1}) = q_i + A_k^{\zeta(j,O)} \quad \text{and} \quad C^{[t_k,t_{k+1}](i,j)}(t_{k+1}) = C_k^{\zeta(j,O)}. \tag{5.15}$$

Proof. Using Theorem 3.1 and the analysis in example 5.2.1 in Yeung and Petrosyan (2006), one can obtain the results in (5.14) and (5.15). □

The solution time paths $A^{[t_k,t_{k+1}](i,j)}(t)$ and $C^{[t_k,t_{k+1}](i,j)}(t)$ for the system of first order differential equations in (44)-(45) can be computed numerically for given values of the model parameters r, q_1, q_2, c_1, c_2, a, b, λ_1, λ_2, and $\varpi_h^{(j,\ell)}$ for $h \in \{1, 2, 3\}$ and $j, \ell \in \{1, 2\}$.

Following Yeung and Petrosyan (2006) the optimal cooperative controls can then be obtained as:

$$\psi_{k-1}^{(i,O)j}(t,x) = \frac{x}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}, \quad \text{and}$$
$$\psi_k^{(j,Y)i}(t,x) = \frac{x}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}. \tag{5.16}$$

Substituting these control strategies into (5.2) yields the dynamics of the state trajectory under cooperation. The optimal cooperative state trajectory in the time interval $[t_k, t_{k+1})$can be obtained as:

$$\frac{dx(s)}{ds} = \Bigg(ax(s)^{1/2} - bx(s) - \frac{x}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(s)/2\right]^2} - \frac{x}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(s)/2\right]^2} \Bigg) ds + \sigma x(s)dz(s), \quad x(t_1) = x_0, \tag{5.17}$$

for $s \in [t_k, t_{k+1})$ and $k \in \{1,2,3\}$.

We denote the set of realizable states at time t from (5.17) under the scenarios of different players by $X_t^{\{t_k,t_{k+1}](i,j)*}$, for $t \in [t_k, t_{k+1})$ and $k \in \{1,2,3\}$. We use the term $x_t^{\{t_k,t_{k+1}](i,j)*} \in X_t^{\{t_k,t_{k+1}](i,j)*}$ to denote an element in $X_t^{\{t_k,t_{k+1}](i,j)*}$. The term x_t^* is used to denote $x_t^{\{t_k,t_{k+1}](i,j)*}$ whenever there is no ambiguity.

5.3. Subgame Consistent Payoff Distribution

According to the solution optimality principle the players agree to share their cooperative payoff according to the solution imputations:

$$\xi^{k-1(i,O)j[\ell]}(t,x) = V^{k-1(i,O)j}(t,x) + w_{k-1}^h[W^{[t_k,t_{k+1}](i,j)}(t,x) - V^{k-1(i,O)j}(t,x) - V^{k(j,Y)i}(t,x)],$$

$$\xi^{k(j,Y)i[\ell]}(t,x) = V^{k(j,Y)i}(t,x) + w_k^h[W^{[t_k,t_{k+1}](i,j)}(t,x) - V^{k-1(i,O)j}(t,x) - V^{k(j,Y)i}(t,x)],$$

for $\ell \in \{1,2,3\}$, $i,j \in \{1,2\}$ and $k \in \{1,2,3\}$.

These imputations are continuous differentiable in x and t. If an imputation vector $[\xi^{k-1(i,O)j[\ell]}(t,x),\xi^{k(j,Y)i[\ell]}(t,x)]$ is chosen, a crucial process is to derive a payoff distribution procedure (PDP) so that this imputation could be realized for $t \in [t_k, t_{k+1})$ along the cooperative trajectory $\{x_t^*\}_{t=t_k}^{t_{k+1}}$.

Following Theorem 4.1, a PDP leading to the realization of the imputation vector $[\xi^{k-1(i,O)j[\ell]}(t,x),\xi^{k(j,Y)i[\ell]}(t,x)]$ can be obtained as:

Corollary 5.1. *A PDP with an instantaneous payment at time $t \in [t_k, t_{k+1})$:*

$$B_{k-1}^{(i,O)j[\ell]}(t) = -\xi_t^{k-1(i,O)j[\ell]}(t,x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t,x_t^*)\xi_{x^h x^\zeta}^{k-1(i,O)j[\ell]}(t,x_t^*)$$
$$-\xi_x^{k-1(i,O)j[\ell]}(t,x_t^*)\Bigg[a(x_t^*)^{1/2} - bx_t^*$$

$$-\frac{x_t^*}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} - \frac{x_t^*}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}\ \Bigg], \qquad (5.18)$$

allocated to the type i generation $k-1$ player;
and an instantaneous payment at time $t \in [t_k, t_{k+1})$:

$$B_k^{(j,Y)i[\ell]}(t) = -\xi_t^{k(j,Y)i[\ell]}(t, x_t^*) - \frac{1}{2}\sum_{h,\zeta=1}^{m} \Omega^{h\zeta}(t, x_t^*)\xi_{x^h x^\zeta}^{k(j,Y)i[\ell]}(t, x_t^*)$$

$$-\xi_x^{k(j,Y)i[\ell]}(t, x_t^*)\Bigg[\ a(x_t^*)^{1/2} - bx_t^*$$

$$-\frac{x_t^*}{4\left[c_i + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2} - \frac{x_t^*}{4\left[c_j + A^{[t_k,t_{k+1}](i,j)}(t)/2\right]^2}\ \Bigg] \qquad (5.19)$$

allocated to the type j generation k player,
yields a mechanism leading to the realization of the imputation vector $[\xi^{k-1(i,O)j[\ell]}(t,x), \xi^{k(j,Y)i[\ell]}(t,x)]$*, for* $k \in \{1,2,3\}$*,* $\ell \in \{1,2,3\}$ *and* $i,j \in \{1,2\}$.

Since the imputations $\xi^{k-1(i,O)j[\ell]}(t,x)$ and $\xi^{k(j,Y)i[\ell]}(t,x)$ are in terms of explicit differentiable functions, the relevant derivatives in Corollary 5.1 can be derived using the results in Propositions 5.1, 5.2 and 5.3. Hence, the PDP $B_{k-1}^{(i,O)j[\ell]}(t)$and $B_k^{(j,Y)i[\ell]}(t)$ in (5.18) and (5.19) can be obtained explicitly.

6. Concluding Remarks and Extensions

This paper considers cooperative stochastic differential games in which players enter the game at different times and have diverse horizons. Moreover, the types of future players are not known with certainty. Subgame consistent cooperative solutions and analytically tractable payoff distribution mechanisms leading to the realization of these solutions are derived. The analysis extends the Yeung (2011) analysis with the incorporation of stochastic dynamics.

The asynchronous horizons game presented can be extended in a couple of directions. First, more complicated stochastic processes can be adopted in the analysis. For instance, the random variable governing the types of future players can be a series of non-identical random variables $\omega_{a_k}^k \in \{\omega_1^k, \omega_2^k, \cdots, \omega_{\varsigma_k}^k\}$ with probabilities $\lambda_{a_k}^k \in \{\lambda_1^k, \lambda_2^k, \cdots, \lambda_{\varsigma_k}^k\}$, for $k \in \{2,3,\cdots,\upsilon\}$.

Secondly, the overlapping generations of players can be extended to more complex structures. The game horizon of the players can include more than two time intervals and be different across players. The number of players in each time interval can also be more than two and be different across intervals. The analysis can be formulated as a general class of stochastic differential games with asynchronous horizons structure. In particular, the type ω_{a_k} generation k player's game horizon is $[t_k, t_{k+\eta_k})$, where $\eta_k \geq 1$. The term $u_k^{(\omega_k,S1)}(s)$ is used to denote the vector of controls of the type ω_{a_k} generation k player in his first game interval $[t_k, t_{k+1})$; and

$u_k^{(\omega_k,S2)}(s)$ is that in his second game interval $[t_{k+1}, t_{k+2})$ and so on. This results in a general class of stochastic differential games with asynchronous horizons structure. Theorem 3.1 and Theorem 4.1 can be readily extended to this general structure with more than two players in each time interval.

Finally, this is the first time that subgame consistent cooperative solutions are analyzed and derived in stochastic differential games with asynchronous players' horizons, further research along this line is expected.

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Subgame Consistent Solution for a Cooperative Differential Game of Climate Change Control

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Abstract After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Moreover, it is now apparent that human activities are perturbing the climate system at the global scale leading to disturbances to complex ecological processes. In this paper, we present a cooperative differential game of climate change control. Climate change is incorporated as structural changes in the pollution dynamics and the payoff functions. The policy instruments of the game include taxes, abatement efforts and production technologies choices. Under cooperation, nations will make use of these instruments to maximize their joint payoff and distribute the payoff according to an agreed upon optimality principle. To ensure that the cooperative solution is dynamically consistent, this optimality principle has to be maintained throughout the period of cooperation. An analytically tractable payment distribution mechanism leading to the realization of the agreed upon imputation is formulated. This analysis widens the application of cooperative differential game theory to environmental problems with climate change. This is also the first time differential games with random changes in the structure of their state dynamics.

Keywords: Cooperative differential games, subgame consistency, climate change, environmental management.

1. Introduction

After several decades of rapid technological advancement and economic growth, alarming levels of pollutions and environmental degradation are emerging all over the world. Moreover, it is now apparent that human activities are perturbing the climate system at the global scale leading to disturbances to complex ecological processes. Climate change is typically structural change that affects the regeneration capacity of the natural environment. Even draconian measures (like a virtual phase-out of fossil fuel) would only slow or stop and not reverse climate change. Reports are portraying the situation as an industrial civilization on the verge of suicide, destroying its environmental conditions of existence with people being held as prisoners on a runaway catastrophe-bound train. Due to the geographical diffusion of pollutants and the global nature of climate change, unilateral response on the part of one country or region is often ineffective. Though cooperation in environmental control holds out the best promise of effective action, limited success has been observed. Existing multinational joint initiatives like the Kyoto Protocol can hardly be expected to offer a long-term solution because (i) the plans are limited only to emissions reduction which is unlikely be able to offer an effective mean to halt the accelerating trend of environmental deterioration brought about by climate change, and (ii) there is no guarantee that participants will always be better off and hence be committed within the entire duration of the agreement.

Differential games provide an effective tool to study pollution control problems and to analyze the interactions between the participants' strategic behaviors and dynamic evolution of pollution. Applications of noncooperative differential games in environmental studies can be found in Yeung (1992), Dockner and Long (1993), Tahvonen (1994), Stimming (1999), Feenstra et al (2001) and Dockner and Leitmann (2001). Cooperative differential games in environmental control are presented by Dockner and Long (1993), Jørgensen and Zaccour (2001), Fredj et al (2004), Breton et al (2005 and 2006), Petrosyan and Zaccour (2003), Yeung (2007) and Yeung and Petrosyan (2008).

In dynamic cooperative games, a credible cooperative agreement has to be dynamically consistent. For dynamic consistency to hold in deterministic games, a stringent condition on the cooperative agreement is required: The specific optimality principle must remain in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. This condition is commonly known as time consistency. In the presence of stochastic elements, a more stringent condition – subgame consistency – is required for a dynamically consistent cooperative solution. A cooperative solution is subgame consistent if an extension of the solution policy to a situation with a later starting time and any feasible state brought about by prior optimal behaviors would remain optimal. Cooperative differential games that have identified dynamically consistent solutions can be found in Jørgensen and Zaccour (2001), Petrosyan and Zaccour (2003), Yeung and Petrosyan (2006a), Yeung (2007), and Yeung and Petrosyan (2004, 2005, 2006b and 2008)).

In this paper, we present a cooperative differential game of climate change control. Climate change is incorporated as structural changes in the pollution dynamics and the payoff functions. Since uncertainties in climate change have been observed (see Berliner (2003) and Allen et al. (2000)), a stochastic formulation of the changes is adopted. The policy instruments available include taxes, abatement efforts and production technologies choices. Under cooperation, nations will make use of these instruments to maximize their joint payoff and distribute the payoff according to an agreed upon optimality principle. To ensure that the cooperative solution is dynamically consistent, this optimality principle has to be maintained throughout the period of cooperation. Crucial to the analysis is the formulation of a payment distribution mechanism so that the agreed upon imputation will be realized. We follow Yeung and Petrosyan (2004 and 2006a) and derive an analytically tractable payment distribution mechanism ensuring the realization of dynamically consistent solutions. This analysis widens the application of cooperative differential game theory to environmental problems with climate change. This is also the first time differential games with randomly changes in the structure of their state dynamics.

The paper is organized as follows. Section 2 provides a game model with technologies choice and climate change. Noncooperative outcomes are characterized in Section 3. Cooperative arrangements, group optimal actions, solution state trajectories, and individually rational and dynamically consistent imputations are examined in Section 4. A payment distribution mechanism bringing about the proposed dynamically consistent solution is derived and scrutinized in Section 5. Section 6 examines the case where partial adoption of climate-preserving technologies appears. Concluding remarks are given in Section 7 and mathematical proofs are provided in the appendices.

2. A Game Model with Technology Choice and Climate Change

In this section we present a differential game model with technology choice and climate change. There are n asymmetric nations (or regions) and the game horizon is $[t_0, T]$.

2.1. The Industrial Sector

These n asymmetric nations form an international or global economy. At time instant s the demand function of the output of nation $i \in N \equiv \{1, 2, \cdots, n\}$ is

$$P_i(s) = \alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s), \tag{2.1}$$

where $P_i(s)$ is the price of the output of nation i, $q_j(s)$ is the output of nation j, α^i and β_j^i for $i \in N$ and $j \in N$ are positive constants. The output choice $q_j(s) \in [0, \bar{q}_j]$ is nonnegative and bounded by a maximum output constraint $\bar{q}_j$. Output price equals zero if the right-hand-side of (2.1) becomes negative. The demand system (2.1) shows that the world economy is a form of differentiated products oligopoly with substitute goods. In the case when $\alpha^i = \alpha^j$ and $\beta_j^i = \beta_i^j$ for all $i \in N$ and $j \in N$, the industrial output is a homogeneous good. This type of model was first introduced by Dixit (1979) and later used in analyses in industrial organizations (see for example, Singh and Vives (1984)) and environmental games (see for examples, Yeung (2007) and Yeung and Petrosyan (2008)).

There are two types of technologies available to each nation's industrial sector: the existing technologies (which is not climate-preserving) and climate-preserving technologies. Industrial sectors pay more for using climate-preserving technologies. The amount of pollutants emitted by climate-preserving technologies is less than that by existing technologies. Moreover, non-climate-preserving technologies do not only emit more pollutants, they also damage the environment like destroying forests, making marine and animal species extinct directly or indirectly, breaking food-chains and desertification. These damages lead to structural climate changes. Therefore it is not pollution *per se* but the use of non-climate-preserving technologies that contributes to climate change. Climate conditions will be preserved only when climate-preserving technologies are used in the economy.

We use $q_j(s)$ to denote the output of nation j produced with existing technologies and $\hat{q}_j(s)$ to denote the output of nation j produced with climate-preserving technologies. The cost of producing $q_j(s)$ units of output with existing technologies is $c_i q_i(s)$ while that of producing $\hat{q}_j(s)$ units of output with climate-preserving technologies is $\hat{c}_i \hat{q}_i(s)$. In addition, $\hat{c}_i > c_i$. In the absence of government regulation or incentive, the industrial sectors will not adopt climate-preserving technologies. In the case when all industrial sectors are using existing technologies industrial profits of nation i at time s can be expressed as:

$$\pi_i(s) = [\alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s)]q_i(s) - c_i q_i(s) - v_i(s) q_i(s), \quad \text{for} \quad i \in N. \tag{2.2}$$

where $v_i(s)$ is the tax rate imposed by government i on industrial output produced by existing technologies at time s. At each time instant s, the industrial sector of nation $i \in N$ seeks to maximize (2.2). The first order condition for a Nash equilibrium for the n nations economy at time s yields

$$\sum_{j=1}^{n} \beta_j^i q_j(s) + \beta_i^i q_i(s) = \alpha^i - c_i - v_i(s), \quad \text{for} \quad i \in N. \tag{2.3}$$

Equation system (2.3) is linear in $q(s) = \{q_1(s), q_2(s), \cdots, q_n(s)\}$. Taking the set of output tax rates $v(s) = \{v_1(s), v_2(s), \cdots, v_n(s)\}$ as parameters and solving (2.3) yield an industrial equilibrium which can be expressed as:

$$q_i(s) = \bar{\alpha}^i + \sum_{j=1}^{n} \bar{\beta}_j^i \, v_j(s), \tag{2.4}$$

where $\bar{\alpha}^i$ and $\bar{\beta}_j^i$, for $i \in N$ and $j \in N$, are constants involving the model parameters $\beta = \{\beta_1^1, \beta_2^1, \cdots, \beta_n^1; \beta_1^2, \beta_2^2, \cdots, \beta_n^2; \cdots; \beta_1^n, \beta_2^n, \cdots, \beta_n^n\}$, $\alpha = \{\alpha^1, \alpha^2, \cdots, \alpha^n\}$ and $c = \{c_1, c_2, \cdots, c_n\}$. Proper choice of parameters leading to a valid industrial equilibrium is assumed.

In the case when all industrial sectors are using climate-preserving technologies industrial profits of nation i at time s can be expressed as:

$$\hat{\pi}_i(s) = [\alpha^i - \sum_{j=1}^{n} \beta_j^i \hat{q}_j(s)]\hat{q}_i(s) - \hat{c}_i \hat{q}_i(s) - v_i(s)\hat{q}_i(s), \quad \text{for} \quad i \in N.$$

Once again a system of linear equations in $\hat{q}(s) = \{\hat{q}_1(s), \hat{q}_2(s), \cdots, \hat{q}_n(s)\}$ is formed. The set of output tax rates $v(s) = \{v_1(s), v_2(s), \cdots, v_n(s)\}$ can be regarded as a set of parameters. An industrial equilibrium gives:

$$\hat{q}_i(s) = \hat{\alpha}^i + \sum_{j=1}^{n} \hat{\beta}_j^i \, v_j(s), \quad \text{for} \quad i \in N.$$

2.2. Pollution Dynamics

Industrial production emits pollutants into the environment and the amount of pollution created by different nations' outputs may be different. Each government adopts its own pollution abatement policy to reduce pollutants existing in the environment. At time t_0 the climate condition in the time interval $[t_0, t_1)$, for $t_1 < T$, is known to be θ_0^0. Let $x(s) \subset R^+$ denote the level of pollution at time s, the dynamics of pollution stock under climate condition θ_0^0 is governed by the differential equation:

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} q_j(s) - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(s)[x(s)]^{1/2} - \delta_{\theta_0^0}\, x(s), \quad x(t_0) = x_{t_0}, \quad s \in [t_0, t_1), \tag{2.5}$$

where $a_j^{\theta_0^0}$ is the amount of pollution created by a unit of nation j's output,

$u_j(s)$ is the level of pollution abatement activities of nation j,

$b_j^{\theta_0^0} u_j(s)[x(s)]^{1/2}$ is the amount of pollution removed by $u_j(s)$ level of abatement activities of nation j,

$\delta_{\theta_0^0}$ is the natural rate of decay of the pollutants, and the initial level of pollution at time t_0 is given as x_{t_0}.

Since existing technologies are not climate preserving, the climate condition will deteriorate. Moreover, uncertainties in climate change have been observed (see Berliner (2003) and Allen et al (2000)). In particular, in future instant of time t_k, for $k \in \{1,\ 2,\ \cdots,\ \tau\}$ and $t_1 < t_2 < \cdots < t_\rho < T \equiv t_{\rho+1}$, the change in climate is affected by a series of random climate variables $\theta_{a_k}^{k[.]}$. If $\theta_{a_k}^{k[.]}$ has occurred at time t_k, it will prevail in the period $[t_k, t_{k+1})$. The process $\theta_{a_k}^{k[.]}$, for $k \in \{1, 2, \cdots, \tau\}$, is a random variable stemming from a randomly branching process as described below.

Given θ_0^0 has occurred in the interval $[t_0, t_1)$, the random variable $\theta_{a_i}^{1[0,a_0]} \in \{\theta_1^{1[0,a_0]}, \theta_2^{1[0,a_0]},\ ..., \theta_{\eta_1}^{1[0,a_0]}\}$ will occur with corresponding probabilities $\{\lambda_1^{1[0,a_0]}, \lambda_2^{1[0,a_0]},\ ..., \lambda_{\eta_1}^{1[0,a_0]}\}$ in the period $[t_1, t_2)$. Note that probabilities of the occurrence of the climate variables are affected by the level of pollution. Given that $\theta_{a_i}^{1[0,a_0]} \in \{\theta_1^{1[0,a_0]}, \theta_2^{1[0,a_0]},\ ..., \theta_{\eta_1}^{1[0,a_0]}\}$ has been realized in $[t_1, t_2)$, the random variable $\theta_{a_2}^{2[(1,a_1)]} \in \{\theta_1^{2[(1,a_1)]}, \theta_2^{2[(1,a_1)]},\ ..., \theta_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]}\}$ will occur with corresponding probabilities $\{\lambda_1^{2[(1,a_1)]}, \lambda_2^{2[(1,a_1)]},\ ..., \lambda_{\eta_{2[(1,a_1)]}}^{2[(1,a_1)]}\}$ in the period $[t_2, t_3)$.

In general, given that $\theta_{a_i}^{1[0,a_0]}, \theta_{a_2}^{2[(1,a_1)]}, \cdots, \theta_{a_{k-1}}^{k-1[(1,a_1)(2,a_2)\ldots(k-2,a_{k-2})]}$ has been realized, the random variable $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]} \in \{\theta_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \theta_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]},\ \ldots \ldots, \theta_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}\}$ will occur with corresponding probabilities $\{\lambda_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \lambda_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \ldots, \lambda_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}\}$ in the period $[t_k, t_{k+1})$, for $k \in \{1, 2, \cdots, \rho\}$. Finally, given that $\theta_{a_i}^{1[0,a_0]}, \theta_{a_2}^{2[(1,a_1)]} \cdots, \theta_{a_\tau}^{\tau[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$ has been realized, the random variable $\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]} \in \{\theta_1^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \theta_2^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \cdots \cdots, \theta_n^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}\}$ will occur with corresponding probabilities $\{\lambda_1^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \lambda_2^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}, \cdots, \lambda_{\eta_{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}\}$ at time T. Irreversible climate change implies that there is no possibility for climate condition to revert to a better state.

If the climate condition $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in the time interval $[t_k, t_{k+1})$, the dynamics of pollution stock becomes:

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} q_j(s) - \sum_{j=1}^{n} b_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} u_j(s)[x(s)]^{1/2}$$
$$- \delta_{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}\, x(s), \quad \text{for} \quad s \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{1, 2, \cdots, \rho\}. \tag{2.6}$$

Dynamics (2.6) reflects that climate change is a structural change such that the transformation of industrial emission into pollutants, the natural rate of decay and the effects of abatement activities could be affected. More specifically, $a_j^{\theta_\varsigma^{k[\cdot]}} > a_j^{\theta_\ell^{k[\cdot]}}$,

$b_j^{\theta_\varsigma^{k[\cdot]}} < b_j^{\theta_\ell^{k[\cdot]}}$ and $\delta_j^{\theta_\varsigma^{k[\cdot]}} < \delta_j^{\theta_\ell^{k[\cdot]}}$ if $\theta_\varsigma^{k[\cdot]}$ represents a worse climate condition than that represented by $\theta_\ell^{k[\cdot]}$.

2.3. The Governments' Objectives

The governments have to promote business interests and at the same time bear the costs brought about by pollution and climate conditions. In particular, each government maximizes the gains in the industrial sector plus tax revenue minus expenditures on pollution abatement, damages from pollution and losses from climate conditions. In the time interval $[t_0, t_1)$ the instantaneous objective of government i can be expressed as:

$$[\alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s)]q_i(s) - c_i[q_i(s)]^2 - c_i^a[u_i(s)]^2 - h_i^{\theta_0^0} x(s) - \varepsilon_i^{\theta_0^0}, \quad i \in N, \tag{2.7}$$

where $c_i^a[u_i(s)]^2$ is the cost of carrying out u_i level of pollution abatement activities, $h_i^{\theta_0^0} x(s)$ is the value of damage to nation i from $x(s)$ amount of pollution, and $\varepsilon_i^{\theta_0^0}$ is cost to nation i under climate condition θ_0^0. Note that the damage from pollution could be related to the climate condition. Moreover, the cost $\varepsilon_i^{\theta_0^0}$ reflects losses from floods, draughts, abnormal temperatures, storms, heat waves, cold spell and similar climate change related problems.

If the climate condition $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}$ occurs in the time interval $[t_k, t_{k+1})$, for $k \in \{1, 2, \cdots, \rho\}$, the instantaneous objective of government i becomes:

$$[\alpha^i - \sum_{j=1}^{n} \beta_j^i q_j(s)]q_i(s) - c_i[q_i(s)] - c_i^a[u_i(s)]^2 - h_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}} x(s)$$
$$- \varepsilon_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}}, \quad i \in N. \tag{2.8}$$

In particular, $h_i^{\theta_\varsigma^{k[\cdot]}} > h_i^{\theta_\ell^{k[\cdot]}}$ and $\varepsilon_i^{\theta_\varsigma^{k[\cdot]}} > \varepsilon_i^{\theta_\ell^{k[\cdot]}}$ if $\theta_\varsigma^{k[\cdot]}$ represents a worse climate condition than that represented by $\theta_\ell^{k[\cdot]}$.

At time T, if $\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}$ occurs, the terminal appraisal of pollution damage is $g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}[\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} - x(T)]$ where $g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} \geq 0$. In particular, if the terminal level of pollution is lower (higher) than $\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}$, government i will receive a bonus (penalty) equaling $g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}[\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} - x(T)]$. Moreover, $g^i_{\theta_\varsigma^{T[\cdot]}} > g^i_{\theta_\ell^{T[\cdot]}}$ and $\bar{x}^i_{\theta_\varsigma^{T[\cdot]}} < \bar{x}^i_{\theta_\ell^{T[\cdot]}}$ if $\theta_\varsigma^{T[\cdot]}$ represents a worse climate condition than that represented by $\theta_\ell^{T[\cdot]}$.

The discount rate is r. Each one of the n governments seeks to maximize the integral of its instantaneous objective specified in (2.7)-(2.8) over the planning horizon subject to pollution dynamics (2.5)-(2.6). By substituting $q_i(s) = \bar{\alpha}^i + \sum_{j=1}^{n} \bar{\beta}_j^i \; v_j(s)$, for $i \in N$, from (2.4) into (2.5)-(2.8), one obtains a stochastic differential game in which government $i \in N$ seeks to:

$$\max_{v_i(s),u_i(s)} \Bigg\{ \int_{t_0}^{t_1} \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s))$$

$$- c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s)] - c_i^a [u_i(s)]^2 - h_i^{\theta_0^0} x(s) - \varepsilon_i^{\theta_0^0} \Bigg] e^{-r(s-t_0)} ds$$

$$+ \sum_{k=1}^{\rho} \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^{1[0,a_0]} \sum_{a_2=1}^{\eta_{2(1,a_1)}} \lambda_{a_2}^{2[1,a_1]} \cdots \sum_{a_k=1}^{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}} \lambda_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$$

$$\times \int_{t_k}^{t_{k+1}} \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s)) - c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s)] -$$

$$c_i^a [u_i(s)]^2 - h_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} x(s) - \varepsilon_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \Bigg] e^{-r(s-t_0)} ds$$

$$+ \sum_{a_1=1}^{\eta_1} \lambda_{a_1}^{1[0,a_0]} \sum_{a_2=1}^{\eta_{2(1,a_1)}} \lambda_{a_2}^{2[1,a_1]} \cdots \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}$$

$$g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} [\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} - x(T)] e^{-r(T-t_0)} \Bigg\}, \quad \text{for} \quad i \in N. \tag{2.9}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s)] - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(s) [x(s)]^{1/2} - \delta_{\theta_0^0} \, x(s),$$

$$x(t_0) = x_{t_0}, \quad s \in [t_0, t_1), \quad \text{and}$$

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} u_j(s) [x(s)]^{1/2}$$

$$- \delta_{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \, x(s), \quad \text{for} \quad s \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{1, 2, \cdots, \rho\}. \tag{2.10}$$

3. Noncooperative Outcomes

In this section we discuss the solution to the noncooperative game (2.9)-(2.10). To obtain a feedback solution for the game, we first consider the solution for the subgame in the last time interval, that is $[t_\rho, T]$. For the case where $\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$ has occurred at time instant t_ρ and $x(t_\rho) = x_{t_\rho} \in X$, player i would seek to:

$$\max_{v_i(s),u_i(s)} \Bigg\{ \int_{t_\rho}^{T} \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s))$$

$$- c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \; v_h(s)] - c_i^a [u_i(s)]^2$$

$$- h_i^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} x(s) - \varepsilon_i^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} \Bigg] e^{-r(s-t_\rho)} ds$$

$$+ \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}} [\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}$$

$$- x(T)] e^{-r(T-t_\rho)} \Bigg\}, \text{for} \quad i \in N, \tag{3.1}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \; v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} u_j(s)[x(s)]^{1/2} - \delta_{\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} x(s),$$

$$x(t_\rho) = x_{t_\rho} \in X \quad \text{for} \quad s \in [t_\rho, T]. \tag{3.2}$$

A feedback Nash equilibrium solution can be characterized with the techniques developed by Isaacs (1965), Bellman (1957) and Nash (1951) as:

Lemma 3.1. *A set of feedback strategies* $\{u_i^*(t) = \mu_i^\rho(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t, x)$, $v_i^*(t) = \phi_i^\rho(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t, x)$, *for* $i \in N$ *and* $t \in [t_\rho, T]\}$ *provides a Nash equilibrium solution to the game (3.11)-(3.12) if there exist continuously differentiable functions* $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t, x)$:$[t_\rho, T] \times R \rightarrow R$, $i \in N$, *satisfying the following partial differential equations:*

$$-V_t^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]}; t, x) = \max_{v_i, u_i} \Bigg\{ \Bigg[[\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^j \; \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]}; t, x) + \bar{\beta}_i^j v_i)]$$

$$\times [\bar{\alpha}^i + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^i \; \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]}; t, x) + \bar{\beta}_i^i v_i] - c_i[\bar{\alpha}^i + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^i \; \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]}; t, x) + \bar{\beta}_i^i v_i]$$

$$- c_i^a [u_i]^2 - h_i^{\theta_{a_\rho}^{\rho[\cdot]}} x - \varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}} \Bigg] e^{-r(t-t_\rho)}$$

$$+V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)\Big[\sum_{j=1}^{n} a_j^{\theta_{a_\rho}^{\rho[\cdot]}}[\bar{\alpha}^j + \sum_{\substack{h=1\\h\neq i}}^{n} \bar{\beta}_h^j\, \phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x) + \bar{\beta}_i^j v_i]$$

$$-\sum_{\substack{j=1\\j\neq i}}^{n} b_j^{\theta_{a_\rho}^{\rho[\cdot]}}\mu_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)x^{1/2} - b_i^{\theta_{a_\rho}^{\rho[\cdot]}}u_i x^{1/2} - \delta_{\theta_{a_\rho}^{\rho[\cdot]}}\, x\ \Big]\Big\},$$

$$V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};T,x) = \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}$$

$$\times[\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} - x(T)]e^{-r(T-t_\rho)};$$

$$for \quad i\in N, \tag{3.3}$$

where $\theta_{a_\rho}^{\rho[\cdot]}$ is the short form for $\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}$.

Performing the indicated maximization in (3.3) yields:

$$\mu_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x) = -\frac{b_i^{\theta_{a_\rho}^{\rho[\cdot]}}}{2c_i^a}V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}x^{1/2},$$

$$\Big(\ \alpha^i - \sum_{j=1}^{n}\beta_j^i[\bar{\alpha}^j + \sum_{h\in N}^{n}\bar{\beta}_h^j\,\phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)]\ \Big)\bar{\beta}_i^i$$

$$-[\sum_{j=1}^{n}\beta_j^i\bar{\beta}_i^j][\bar{\alpha}^i + \sum_{h\in 1}^{n}\bar{\beta}_h^i\,\phi_h^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)] - c_i\bar{\beta}_i^i$$

$$+V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}\sum_{j=1}^{n} a_j^{\theta_{a_\rho}^{\rho[\cdot]}}\bar{\beta}_i^j\ = 0, \tag{3.4}$$

for $t\in[t_\rho < T]$ and $i\in N$.

System (3.4) forms a set of equations linear in $\{\phi_1^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x), \phi_2^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x),\dots,\phi_n^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x)\}$ with

$$\{V_x^{(t_\rho)1}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}, V_x^{(t_\rho)2}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)},\cdots,V_x^{(t_\rho)n}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}\}$$

being taken as a set of parameters. Solving the second set of equations in (3.4) yields:

$$\phi_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x) = \tilde{\alpha}^i_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{j=1}^{n}\tilde{\beta}^i_{\theta_{a_\rho}^{\rho[\cdot]}j}\,V_x^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x)e^{r(t-t_\rho)}, \quad i\in N, \tag{3.5}$$

where $\tilde{\alpha}^i_{\theta_{a_\rho}^{\rho[\cdot]}}$ and $\tilde{\beta}^i_{\theta_{a_\rho}^{\rho[\cdot]}j}$, for $i\in N$ and $j\in N$, are constants involving the constant coefficients in (3.4). Substituting the results in (3.4) and (3.5) into (3.3) and upon solving we obtain:

Proposition 3.1. *System (3.3) admits a solution*

$$V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]};t,x) = [A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x + C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\, e^{-r(t-t_\rho)}, \quad for \quad i \in N, \tag{3.6}$$

where $\{A_1^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), A_2^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), \cdots, A_n^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)\}$ *and* $\{C_1^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), C_2^\rho(\theta_{a_\rho}^{\rho[\cdot]};t), \cdots, C_n^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)\}$ *satisfy the following sets of constant differential equations:*

$$\dot{A}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) = (r + \delta_{\theta_{a_\rho}^{\rho[\cdot]}})\, A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) - \frac{(b_i^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{4c_i^a}[A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t))]^2$$

$$-A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{(b_j^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{2c_j^a} A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) + h_i^{\theta_{a_\rho}^{\rho[\cdot]}}; \tag{3.7}$$

$$\dot{C}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) = rC_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) - \left(\alpha^i - \sum_{j=1}^n \beta_j^i \{\bar\alpha^j + \sum_{h=1}^n \bar\beta_h^j\, [\hat\alpha^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \hat\beta^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} \right)$$

$$\left(\bar\alpha^i + \sum_{h=1} \bar\beta_h^i\, [\tilde\alpha^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde\beta^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)] \right)$$

$$+c_i\{\bar\alpha^i - \sum_{j=1}^n \bar\beta_j^i\, [\tilde\alpha^j_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde\beta^j_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} + \varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}}$$

$$-A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \left[\sum_{j=1}^n a_j^{\theta_{a_\rho}^{\rho[\cdot]}} \{\bar\alpha^j + \sum_{h=1}^n \bar\beta_h^j\, [\tilde\alpha^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde\beta^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\, A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} \right]; \tag{3.8}$$

$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = -\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}$ *and*

$$C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}$$

$$\times g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} \bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}};$$

$$for \quad i \in N. \tag{3.9}$$

Proof. See Appendix 1. □

Using (3.4), (3.5) and the results in Proposition 3.1, the corresponding feedback Nash equilibrium strategies of the game (3.1)-(3.2) can be obtained as:

$$\mu_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t,x) = -\frac{b_i^{\theta_{a_\rho}^{\rho[\cdot]}}}{2c_i^a} A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x^{1/2}, \quad \text{and}$$

$$\phi_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]}; t, x) = \hat{\alpha}^i_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{j=1}^{n} \hat{\beta}^i_{\theta_{a_\rho}^{\rho[\cdot]} j} A_j^{\rho}(\theta_{a_\rho}^{\rho[\cdot]}; t), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_\tau, T].$$

A remark that will be utilized in subsequent analysis is given below.

Remark 3.1. Let $V^{\rho(\tau)i}(\theta_{a_\rho}^{\rho[\cdot]}; t, x)$ denote the value function of nation i in a game with payoffs (3.1) and dynamics (3.2) which starts at time τ for $\tau \in [t_\rho, T]$. One can readily verify that $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]}; t, x) = e^{r(\tau - t_\rho)} V^{\rho(\tau)i}(\theta_{a_\rho}^{\rho[\cdot]}; t, x)$, for $\tau \in [t_\rho, T]$.

Lemma 3.1 characterizes the players' value function $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[\cdot]}; t, x)$ during the time interval $[t_\rho, T]$ in the case where $\theta_{a_\rho}^{\rho[\cdot]} \in \{\theta_1^{\rho[\cdot]}, \theta_2^{\rho[\cdot]}, \cdots, \theta_{\eta_\rho}^{\rho[\cdot]}\}$ has occurred. In order to formulate the subgame in the second last time interval $[t_{\rho-1}, t_\rho)$, it is necessary to identify the terminal payoffs at time t_ρ. To do this, first note that if $\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$ occurs at time t_ρ the value function of player i is $V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t_\rho, x)$ at t_ρ. The expected terminal payoff for player i at time t_ρ can evaluated as:

$$\sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]} V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t_\rho, x),$$

$$\text{for} \quad i \in N, \tag{3.10}$$

For the case where $\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}$ occurs in time interval $[t_{\rho-1}, t_\rho)$ and $x(t_{\rho-1}) = x_{\rho-1}$ at time $t_{\rho-1}$, the subgame in question becomes an $n-$person game with duration $[t_{\rho-1}, t_\rho)$, in which player i maximizes the expected payoff:

$$\int_{t_{\rho-1}}^{t_\rho} \Bigg[\; [\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s))$$

$$-c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s)] - c_i^a [u_i(s)]^2 - h_i^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} x(s)$$

$$-\varepsilon_i^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} \Bigg] e^{-r(s-t_{\rho-1})} ds$$

$$+ \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\ldots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}$$

$$\times V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\ldots(\rho-1,a_{\rho-1})]}; t_\rho, x) e^{-r(t_\rho - t_{\rho-1})},$$

$$\text{for} \quad i \in N, \tag{3.11}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} u_j(s)[x(s)]^{1/2}$$

$$-\delta_{\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}} x(s), \quad x(t_{\tau-1}) = x_{t_{\rho-1}} \in X, \quad \text{for} \quad s \in [t_{\rho-1}, t_\rho). \tag{3.12}$$

A feedback Nash equilibrium solution can be characterized as:

Lemma 3.2. *A set of feedback strategies* $\{u_i^*(t) = \mu_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}; t, x),\ v_i^*(t) = \phi_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}; t, x)$, *for* $i \in N$ *and* $t \in [t_\tau, T]\}$ *provides a Nash equilibrium solution to the game (3.11)-(3.12) if there exist continuously differentiable functions* $V^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\ldots(\rho-2,a_{\rho-2})]}; t, x) : [t_{\rho-1}, t_\rho] \times R \rightarrow R$, $i \in N$, *satisfying the following partial differential equations:*

$$-V_t^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) = \max_{v_i, u_i} \Bigg\{ \Bigg[\ [\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^j \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) + \bar{\beta}_i^j v_i)]$$

$$\times [\bar{\alpha}^i + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^i \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) + \bar{\beta}_i^i v_i]$$

$$-c_i[\bar{\alpha}^i + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^i \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) + \bar{\beta}_i^i v_i] - c_i^a [u_i]^2 - h_i^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} x - \varepsilon_i^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} \ \Bigg] e^{-r(t - t_{\rho-1})}$$

$$+V_x^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) \Bigg[\ \sum_{j=1}^{n} a_j^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} [\bar{\alpha}^j + \sum_{\substack{h=1 \\ h \neq i}}^{n} \bar{\beta}_h^j \, \phi_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) + \bar{\beta}_i^j v_i]$$

$$- \sum_{\substack{j=1 \\ j \neq i}}^{n} b_j^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} \mu_h^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]}; t, x) x^{1/2} - b_i^{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} u_i x^{1/2} - \delta_{\theta_{a_{\rho-1}}^{\rho-1[\cdot]}} x \ \Bigg] \Bigg\},$$

$$V^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t_\rho,x) = \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\dots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}$$

$$\times V^{(t_\rho)i}(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]};t_\rho,x)e^{-r(t_\rho-t_{\rho-1})};$$

for $i \in N$, (3.13)

where $\theta_{a_{\rho-1}}^{\rho-1[\cdot]}$ *is the short form for* $\theta_{a_{\rho-1}}^{\rho-1[(1,a_1)\,(2,a_2)\dots(\rho-2,a_{\rho-2})]}$.

Following the analysis above, the value functions $V^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t,x)$ can be characterized as:

Proposition 3.2.

$$V^{(t_{\rho-1})i}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t,x) = [A_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t)x + C_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t)]\; e^{-r(t-t_{\rho-1})},$$

for $i \in N$, (3.14)

where $\{A_1^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t), A_2^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t), \cdots, A_n^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t)\}$ *and* $\{C_1^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t), C_2^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t), \cdots, C_n^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t)\}$ *satisfy*

$$A_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};T) = \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\dots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}$$

$$\times A_i^\rho(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]};t_\rho,x),$$

$$C_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};T) = \sum_{a_\rho}^{\eta_{\rho[(1,a_1)(2,a_2)\dots(\rho-1,a_{\rho-1})]}} \lambda_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]}$$

$$\times C_i^\rho(\theta_{a_\rho}^{\rho[(1,a_1)\,(2,a_2)\dots(\rho-1,a_{\rho-1})]};t_\rho,x); \tag{3.15}$$

and equations (3.7) and (3.8) with
$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)$ *replaced by* $A_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t)$, $C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)$ *by* $C_i^{\rho-1}(\theta_{a_{\rho-1}}^{\rho-1[\cdot]};t)$, *and* $\theta_{a_\rho}^{\rho[\cdot]}$ *by* $\theta_{a_{\rho-1}}^{\rho-1[\cdot]}$.

Proof. Follow the Proof of Appendix 1. □

Following the above analysis, for the subgame in the interval $[t_k, t_{k+1})$ with $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}$ occurs in interval, the expected terminal payoff for player i at time t_{k+1} can evaluated as:

$$\sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]}$$

$$\times V^{(t_{k+1})i}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]};t_{k+1},x), \quad \text{for} \quad i \in N. \tag{3.16}$$

For the case where $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in time interval $[t_k, t_{k+1})$ and $x(t_k) = x_k$ at time t_k for $k \in \{0, 1, 2, \cdots, \tau - 1\}$, the subgame in question becomes an $n-$person game with duration $[t_k, t_{k+1})$, in which player i maximizes the expected payoff:

$$\int_{t_k}^{t_{k+1}} \Bigg[\; [\alpha^i - \sum_{j=1}^{n} \beta_j^i (\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s))](\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s)) - c_i[\bar{\alpha}^i + \sum_{h=1}^{n} \bar{\beta}_h^i \, v_h(s)]$$

$$-c_i^a [u_i(s)]^2 - h_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} x(s) - \varepsilon_i^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \Bigg] \, e^{-r(s-t_k)} ds$$

$$+ \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$

$$\times V^{(k+1)i}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}; t_{k+1}, x(t_{k+1}))e^{-r(t_{k+1}-t_k)},$$

$$\text{for} \quad i \in N, \tag{3.17}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} [\bar{\alpha}^j + \sum_{h=1}^{n} \bar{\beta}_h^j \, v_h(s)]$$

$$- \sum_{j=1}^{n} b_j^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} u_j(s)[x(s)]^{1/2}$$

$$-\delta_{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}} \, x(s), \quad x(t_k) = x_{t_k} \in X \quad \text{for} \quad s \in [t_k, t_{k+1}). \tag{3.18}$$

Following the above analysis, one can obtain

Proposition 3.3.

$$V^{(t_k)i}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t, x) =$$

$$[A_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)x + C_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)] \; e^{-r(t-t_k)}, \tag{3.19}$$

for $i \in N$, $t \in [t_k, t_{k+1}]$,
$\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]} \in \{\theta_1^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \theta_2^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}, \ldots$
$, \theta_{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}\}$ *and* $k \in \{0, 1, 2, \cdots, \rho - 1\}$;
where $A_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)$ *and* $C_i^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}; t)$, *for* $i \in N$, *satisfy the following sets of differential equations:*

$$\dot{A}_i^k(\theta_{a_k}^{k[\cdot]};t) = (r+\delta_{\theta_{a_k}^{k[\cdot]}})\,A_i^k(\theta_{a_k}^{k[\cdot]};t) - \frac{(b_i^{\theta_{a_k}^{k[\cdot]}})^2}{4c_i^a}[A_i^k(\theta_{a_k}^{k[\cdot]};t))]^2$$

$$-A_i^k(\theta_{a_k}^{k[\cdot]};t)\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{(b_j^{\theta_{a_k}^{k[\cdot]}})^2}{2c_j^a}A_i^k(\theta_{a_k}^{k[\cdot]};t) + h_i^{\theta_{a_k}^{k[\cdot]}}\,; \tag{3.20}$$

$$\dot{C}_i^k(\theta_{a_k}^{k[\cdot]};t) \;= rC_i^k(\theta_{a_k}^{k[\cdot]};t)$$

$$-\left(\alpha^i - \sum_{j=1}^{n}\beta_j^i\{\bar{\alpha}^j + \sum_{h=1}^{n}\bar{\beta}_h^j\,[\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n}\tilde{\beta}^h_{\theta_{a_k}^{k[\cdot]}\ell}\,A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\}\right)$$

$$\left(\bar{\alpha}^i + \sum_{h=1}^{n}\bar{\beta}_h^i\,[\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n}\tilde{\beta}^h_{\theta_{a_k}^{k[\cdot]}\ell}\,A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\right)$$

$$+c_i\{\bar{\alpha}^i - \sum_{j=1}^{n}\bar{\beta}_j^i\,[\tilde{\alpha}^j_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n}\tilde{\beta}^j_{\theta_{a_k}^{k[\cdot]}\ell}\,A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\} + \varepsilon_i^{\theta_{a_k}^{k[\cdot]}}$$

$$-A_i^k(\theta_{a_k}^{k[\cdot]};t)\left[\sum_{j=1}^{n}a_j^{\theta_{a_k}^{k[\cdot]}}\{\bar{\alpha}^j + \sum_{h=1}^{n}\bar{\beta}_h^j\,[\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \sum_{\ell=1}^{n}\tilde{\beta}^h_{\theta_{a_k}^{k[\cdot]}\ell}\,A_\ell^k(\theta_{a_k}^{k[\cdot]};t)]\}\right];$$

$$A_i^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}}\lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]}$$

$$\times A_i^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]};t_{k+1},x),$$

$$C_i^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\dots(k,a_k)]}}\lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]}$$

$$\times C_i^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\dots(k,a_k)]};t_{k+1},x); \tag{3.21}$$

where $\theta_{a_k}^{k[\cdot]}$ *is the short form for* $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\dots(k-1,a_{k-1})]}$.

Proof. Follow the proofs of Propositions 3.1 and 3.2. □

The corresponding feedback Nash equilibrium strategies of the game (3.17)–(3.18) can be obtained as:

$$\mu_i^k(\theta_{a_k}^{k[\cdot]};t,x) = -\frac{b_i^{\theta_{a_k}^{k[\cdot]}}}{2c_i^a}A_i^k(\theta_{a_k}^{k[\cdot]};t)x^{1/2}, \quad \text{and}$$

$$\phi_i^k(\theta_{a_k}^{k[\cdot]};t,x) = \tilde{\alpha}^i_{\theta_{a_k}^{k[\cdot]}} + \sum_{j=1}^{n}\tilde{\beta}^i_{\theta_{a_k}^{k[\cdot]}j}A_j^k(\theta_{a_k}^{k[\cdot]};t), \quad \text{for} \quad i\in N \quad \text{and} \quad t\in[t_k,t_{k+1});$$

where $\theta_{a_k}^{k[\cdot]}$ is the short form for $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$.

Though continual adoption of non-climate-preserving technologies would lead to further irreversible climate deterioration, nations have no incentive to switch to climate-preserving technologies while other nations are using non-climate-preserving technologies. A global ban on non-climate-preserving technologies would unlikely receive unanimous approval because some nations may face higher production cost differentials in switching to climate-preserving technologies than others'. Only through cooperation and proper appropriation of gains could the problem be tackled.

4. Cooperative Arrangements in Climate Change Control

Now consider the case when all the nations want to cooperate and agree to act so that an international optimum could be achieved. Cooperation will cease if any of the nations refuses to act accordingly at any time within the game horizon. An agreement on the choice of technologies, taxes imposed, abatement efforts and an optimality principle to allocate the cooperative payoff will be sought. For the cooperative scheme to be upheld throughout the game horizon both group rationality and individual rationality are required to be satisfied at any time. Group optimality ensures that all potential gains from cooperation are captured. Failure to fulfill group optimality leads to condition where the participants prefer to deviate from the agreed upon solution plan in order to extract the unexploited gains. Individual rationality is required to hold so that the payoff allocated to a nation under cooperation will be no less than its noncooperative payoff. Failure to guarantee individual rationality leads to the condition where the concerned participants would reject the agreed upon solution plan and play noncooperatively.

4.1. Group Optimality and Cooperative State Trajectory

Given that there are two technologies available, the nations have a technology choice. Consider first the case when all nations agree to adopt climate-preserving technologies from time t_0 to time T. If such technologies were used, the climate will be preserved as θ_0^0 throughout the game duration. To secure group optimality the participating nations seek to maximize their joint expected payoff by solving the following control problem:

$$\max_{v_1,v_2,\cdots,v_n;\,u_1,u_2,\cdots,u_n} \Bigg\{ \sum_{\kappa=1}^{n} \int_{t_0}^{T} \Bigg[\; [\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa (\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \; v_h(s))](\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \; v_h(s))$$

$$-\hat{c}_\kappa[\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \; v_h(s)] - c_\kappa^a[u_\kappa(s)]^2 - h_\kappa^{\theta_0^0} x(s) - \varepsilon_\kappa^{\theta_0^0} \; \Bigg] \; e^{-r(s-t_0)} ds$$

$$+ \sum_{\kappa=1}^{n} g_{\theta_0^0}^\kappa [\bar{x}_{\theta_0^0}^\kappa - x(T)] e^{-r(T-t_0)} \; \Bigg\} \tag{4.1}$$

subject to

$$\dot{x}(s) = \sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \; v_h(s)] - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j(s)[x(s)]^{1/2} - \delta_{\theta_0^0} \, x(s), x(t_0) = x_{t_0}. \tag{4.2}$$

Invoking Bellman's (1957) technique of dynamic programming a set of controls $\Big\{ [v_i^{**}(t), u_i^{**}(t)] = [\psi_i^{\theta_0^0}(t,x), \varpi_i^{\theta_0^0}(t,x)]$, for $i \in N \}$ constitutes an optimal solution to the control problem (4.1) and (4.2) if there exists continuously differentiable function $W^{(t_0)}(\theta_0^0; t, x){:}[t_0, T] \times R \to R$, $i \in N$, satisfying the following partial differential equations:

$$-W_t^{(t_0)}(\theta_0^0; t, x) =$$

$$\max_{v_1, v_2, \cdots, v_n; u_1, u_2, \cdots, u_n} \Bigg\{ \sum_{\kappa=1}^{n} \Bigg[\; [\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa (\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \; v_h)](\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \; v_h)$$

$$-\hat{c}_\kappa [\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \; v_h] - c_\kappa^a (u_\kappa)]^2 - h_\kappa^{\theta_0^0} x - \varepsilon_\kappa^{\theta_0^0} \; \Bigg] \, e^{-r(t-t_0)}$$

$$+W_x^{(t_0)}(\theta_0^0; t, x) \left(\sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \; v_h] - \sum_{j=1}^{n} b_j^{\theta_0^0} u_j (x)^{1/2} - \delta_{\theta_0^0} \, x \right),$$

$$W^{(t_0)}(\theta_0^0; T, x) = \sum_{\kappa=1}^{n} g_{\theta_0^0}^\kappa [\bar{x}_{\theta_0^0}^\kappa - x(T)] e^{-r(T-t_0)}. \tag{4.3}$$

Performing the indicated maximization in (4.3) yields the optimal controls under cooperation as:

$$\varpi_i^{\theta_0^0}(t,x) = -\frac{b_i^{\theta_0^0}}{2c_i^a} W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)} x^{1/2}, \quad \text{for} \quad i \in N; \tag{4.4}$$

$$\sum_{\kappa=1}^{n} \Bigg[\; \Bigg(\; \alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \; \psi_h^{\theta_0^0}(t,x)] \; \Bigg) \bar{\beta}_i^\kappa$$

$$-[\sum_{j=1}^{n} \beta_j^\kappa \hat{\beta}_i^j][\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa \; \psi_h^{\theta_0^0}(t,x)] - \hat{c}_\kappa \hat{\beta}_i^\kappa \; \Bigg]$$

$$+W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)} \sum_{j=1}^{n} a_j^{\theta_0^0} \hat{\beta}_i^j = 0, \quad \text{for} \quad i \in N. \tag{4.5}$$

System (4.5) can be viewed as a set of equations linear in $\{\psi_1^{\theta_0^0}(t,x), \psi_2^{\theta_0^0}(t,x), \cdots, \psi_n^{\theta_0^0}(t,x)\}$ with $W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)}$ being taken as a parameter. Solving (**??**) yields:

$$\psi_i^{\theta_0^0}(t,x) = \hat{\hat{\alpha}}_{\theta_0^0}^i + \hat{\hat{\beta}}_{\theta_0^0}^i W_x^{(t_0)}(\theta_0^0; t, x) e^{r(t-t_0)}, \tag{4.6}$$

where $\hat{\hat{\alpha}}_{\theta_0^0}^i$ and $\hat{\hat{\beta}}_{\theta_0^0}^i$, for $i \in N$, are constants involving the parameters in (4.5).

Proposition 4.1. *System (4.3) admits a solution*

$$W^{(t_0)}(\theta_0^0;t,x) = [A^*_{\theta_0^0}(t)x + C^*_{\theta_0^0}(t)]\, e^{-r(t-t_0)}, \tag{4.7}$$

with

$$\dot{A}^*_{\theta_0^0}(t) = (r+\delta_{\theta_0^0})\, A^*_{\theta_0^0}(t) - \sum_{j=1}^n \frac{(b_j^{\theta_0^0})^2}{2c_j^a}[A^*_{\theta_0^0}(t)]^2 + \sum_{j=1}^n h_j^{\theta_0^0},$$

$$\dot{C}^*_{\theta_0^0}(t) = rC^*_{\theta_0^0}(t)$$

$$-\sum_{\kappa=1}^n \Bigg[\left(\alpha^\kappa - \sum_{j=1}^n \beta_j^\kappa \{\hat{\alpha}^j + \sum_{h=1}^n \hat{\beta}_h^j\, [\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\}\right) \{\hat{\alpha}^\kappa$$

$$+ \sum_{h=1}^n \hat{\beta}_h^\kappa\, [\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\} - \hat{c}_\kappa \{\hat{\alpha}^\kappa + \sum_{j=1}^n \hat{\beta}_j^\kappa\, [\hat{\hat{\alpha}}^j_{\theta_0^0} + \hat{\hat{\beta}}^j_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\} - \varepsilon_\kappa^{\theta_0^0} \Bigg]$$

$$-A^*_{\theta_0^0}(t) \Bigg[\sum_{j=1}^n a_j^{\theta_0^0} \{\hat{\alpha}^j + \sum_{h=1}^n \hat{\beta}_h^j\, [\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(t)]\} \Bigg],$$

$A^*_{\theta_0^0}(T) = -\sum_{j=1}^n g^j_{\theta_0^0}$ *and* $C^*_{\theta_0^0}(T) = \sum_{j=1}^n g^j_{\theta_0^0} \bar{x}^j_{\theta_0^0}$.

Proof. See Appendix 2. □

Using (4.4), (4.6) and (4.7), the control strategy under cooperation can be obtained as:

$$\psi_i^{\theta_0^0}(t,x) = \hat{\hat{\alpha}}^i_{\theta_0^0} + \hat{\hat{\beta}}^i_{\theta_0^0}\, A^*_{\theta_0^0}(t) \quad \text{and} \quad \varpi_i^{\theta_0^0}(t,x) = -\frac{b_i^{\theta_0^0}}{2c_i^a} A^*_{\theta_0^0}(t) x^{1/2}, \tag{4.8}$$

for $t \in [t_0 < T]$ and $i = 1, 2, \cdots, n$.

Substituting the optimal control strategy from (4.8) into (4.2) yields the dynamics of pollution accumulation under cooperation as:

$$\dot{x}(s) = \sum_{j=1}^n a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^n \hat{\beta}_h^j\, (\hat{\hat{\alpha}}^h_{\theta_0^0} + \hat{\hat{\beta}}^h_{\theta_0^0}\, A^*_{\theta_0^0}(s))]$$

$$+ \sum_{j=1}^n \frac{(b_j^{\theta_0^0})^2}{2c_j^a} A^*_{\theta_0^0}(s)x(s) - \delta_{\theta_0^0}\, x(s), \quad x(t_0) = x_{t_0}. \tag{4.9}$$

(4.9) is a linear differential equation with time varying coefficients. We use $\{x^*(s)\}_{s=t_0}^T$ to denote the solution path satisfying (4.9). The term x_t^* is used interchangeably with $x^*(t)$.

A remark that will be utilized in subsequent analysis is given below.

Remark 4.1. Let $W^{(\tau)}(\theta_0^0;t,x)$ denote the value function of the control problem with objective (4.1) and dynamics (2.8) which starts at time τ. One can readily verify that $W^{(\tau)}(\theta_0^0;t,x)$=$W^{(t_0)}(\theta_0^0;t,x)e^{r(\tau-t_0)}$, for $\tau\in[t_0,T]$.

Now consider the case when existing technologies are adopted throughout the cooperative scheme. To secure group optimality the participating nations seek to maximize their joint expected payoff by solving the following control problem:

$$\max_{v_1,v_2,\cdots,v_n;\,u_1,u_2,\cdots,u_n}\Bigg\{\sum_{\varsigma=1}^{n}\int_{t_0}^{t_1}\Bigg[\ [\alpha^\varsigma-\sum_{j=1}^{n}\beta_j^\varsigma(\bar\alpha^j+\sum_{h=1}^{n}\bar\beta_h^j\ v_h(s))](\bar\alpha^\varsigma+\sum_{h=1}^{n}\bar\beta_h^\varsigma\ v_h(s))$$

$$-c_\varsigma[\bar\alpha^\varsigma+\sum_{h=1}^{n}\bar\beta_h^\varsigma\ v_h(s)]-c_\varsigma^a[u_\varsigma(s)]^2-h_\varsigma^{\theta_0^0}x(s)-\varepsilon_\varsigma^{\theta_0^0}\ \Bigg]\ e^{-r(s-t_0)}ds$$

$$+\sum_{\varsigma=1}^{n}\sum_{k=1}^{\rho}\sum_{a_1=1}^{\eta_1}\lambda_{a_1}^{1[0,a_0]}\sum_{a_2=1}^{\eta_{2(1,a_1)}}\lambda_{a_2}^{2[1,a_1]}\cdots\sum_{a_k=1}^{\eta_{k[(1,a_1)(2,a_2)\ldots(k-1,a_{k-1})]}}\lambda_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$$

$$\times\int_{t_k}^{t_{k+1}}\Bigg[\ [\alpha^\varsigma-\sum_{j=1}^{n}\beta_j^\varsigma(\bar\alpha^j+\sum_{h=1}^{n}\bar\beta_h^j\ v_h(s))](\bar\alpha^\varsigma+\sum_{h=1}^{n}\bar\beta_h^\varsigma\ v_h(s))$$

$$-c_\varsigma[\bar\alpha^\varsigma+\sum_{h=1}^{n}\bar\beta_h^\varsigma\ v_h(s)]-c_\varsigma^a[u_\varsigma(s)]^2$$

$$-h_\varsigma^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}x(s)-\varepsilon_\varsigma^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}\ \Bigg]\ e^{-r(s-t_0)}ds$$

$$+\sum_{\varsigma=1}^{n}\sum_{a_1=1}^{\eta_1}\lambda_{a_1}^{1[0,a_0]}\sum_{a_2=1}^{\eta_{2(1,a_1)}}\lambda_{a_2}^{2[1,a_1]}\cdots\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\ldots(\rho,a_\rho)]}}\lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}$$

$$g^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}[\bar x^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\rho,a_\rho)]}}-x(T)]e^{-r(T-t_0)}\ \Bigg\},\quad\text{for}\quad i\in N.\quad(4.10)$$

subject to (2.10).

Following the analysis leading to Propositions 3.1, 3.2, 3.3 and 4.1, one can obtain:

Proposition 4.2. *For the case where $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in time interval $[t_k,t_{k+1})$ at time t_k for $k\in\{0,1,2,\cdots,\tau\}$, the value function $\tilde W^{(t_k)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t,x)$ can be expressed as:*

$$\tilde W^{(t_k)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t,x)=$$

$$[A^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)x + C^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)]\; e^{-r(t-t_k)}, \tag{4.11}$$

where $A^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)$ *and* $C^k(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t)$, *for* $i \in N$, *satisfy the following sets of differential equations:*

$$\dot{A}^k(\theta_{a_k}^{k[\cdot]};t) = (r + \delta_{\theta_{a_k}^{k[\cdot]}})\,A^k(\theta_{a_k}^{k[\cdot]};t) - \sum_{j=1}^n \frac{(b_j^{\theta_{a_k}^{k[\cdot]}})^2}{4c_j^a}[A^k(\theta_{a_k}^{k[\cdot]};t))]^2 + \sum_{j=1}^n h_j^{\theta_{a_k}^{k[\cdot]}},$$

$$\dot{C}^k(\theta_{a_k}^{k[\cdot]};t) \;= rC^k(\theta_{a_k}^{k[\cdot]};t)$$

$$-\sum_{\varsigma=1}^n \Bigg[\left(\alpha^\varsigma - \sum_{j=1}^n \beta_j^\varsigma\{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j\;[\tilde{\bar{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^h_{\theta_{a_k}^{k[\cdot]}}A^k(\theta_{a_k}^{k[\cdot]};t)]\}\right)$$

$$\left(\;\bar{\alpha}^\varsigma + \sum_{h=1}^n \bar{\beta}_h^\varsigma\;[\tilde{\bar{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^h_{\theta_{a_k}^{k[\cdot]}}A^k(\theta_{a_k}^{k[\cdot]};t)]\;\right)$$

$$-c_\varsigma\{\bar{\alpha}^\varsigma - \sum_{j=1}^n \bar{\beta}_j^\varsigma\;[\tilde{\bar{\alpha}}^j_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^j_{\theta_{a_k}^{k[\cdot]}}A^k(\theta_{a_k}^{k[\cdot]};t)]\} - \varepsilon_\varsigma^{\theta_{a_k}^{k[\cdot]}}\;\Bigg]$$

$$-A^k(\theta_{a_k}^{k[\cdot]};t)\left[\;\sum_{j=1}^n a_j^{\theta_{a_k}^{k[\cdot]}}\{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j\;[\tilde{\bar{\alpha}}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\bar{\beta}}^h_{\theta_{a_k}^{k[\cdot]}}A^k(\theta_{a_k}^{k[\cdot]};t)]\}\;\right];$$

$$A^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$

$$\times A^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]};t_{k+1},x),$$

$$C^k(\theta_{a_k}^{k[\cdot]};T) = \sum_{a_{k+1}}^{\eta_{k+1[(1,a_1)(2,a_2)\ldots(k,a_k)]}} \lambda_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]}$$

$$\times C^{k+1}(\theta_{a_{k+1}}^{k+1[(1,a_1)\,(2,a_2)\ldots(k,a_k)]};t_{k+1},x);$$

where $\theta_{a_k}^{k[\cdot]}$ *is the short form for* $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$, *and*

$$A^{\tau+1}(\theta_{a_{\tau+1}}^{\tau+1[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]};t_{\tau+1},x) = -\sum_{\varsigma=1}^n g^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]}},$$

$$C^{\tau+1}(\theta_{a_{\tau+1}}^{\tau+1[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]};t_{\tau+1},x) = \sum_{\varsigma=1}^n g^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]}}$$

$$\times \bar{x}^\varsigma_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\ldots(\tau,a_\tau)]}}.$$

□

In the case where $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$ occurs in time interval $[t_k, t_{k+1})$ at time t_k for $k \in \{0,1,2,\cdots,\tau\}$, the optimal control strategy under cooperation can be obtained as:

$$\tilde{\omega}_i^k(\theta_{a_k}^{k[\cdot]};t,x) = -\frac{b_i^{\theta_{a_k}^{k[\cdot]}}}{2c_i^a} A^k(\theta_{a_k}^{k[\cdot]};t)x^{1/2}, \quad \text{and}$$

$$\tilde{\psi}_i^k(\theta_{a_k}^{k[\cdot]};t,x) = \tilde{\alpha}^i_{\theta_{a_k}^{k[\cdot]}} + \tilde{\tilde{\beta}}^i_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]};t), \quad \text{for} \quad i \in N \quad \text{and} \quad t \in [t_k, t_{k+1}); \quad (4.12)$$

where $\theta_{a_k}^{k[\cdot]}$ is the short form for $\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}$, and $\tilde{\alpha}^i_{\theta_{a_k}^{k[\cdot]}}$ and $\tilde{\tilde{\beta}}^i_{\theta_{a_k}^{k[\cdot]}}$ are the counterparts of $\hat{\hat{\alpha}}^i_{\theta_0^0}$ and $\hat{\hat{\beta}}^i_{\theta_0^0}$ in (4.8).

Substituting the optimal control strategy in (4.12) into (2.10) yields the cooperative state trajectory using existing technologies as:

$$\dot{x}(s) = \sum_{j=1}^n a_j^{\theta_{a_k}^{k[\cdot]}} \{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j [\tilde{\alpha}^h_{\theta_{a_k}^{k[\cdot]}} + \tilde{\tilde{\beta}}^h_{\theta_{a_k}^{k[\cdot]}} A^k(\theta_{a_k}^{k[\cdot]};s)]\}$$

$$+ \sum_{j=1}^n \frac{(b_j^{\theta_{a_k}^{k[\cdot]}})^2}{2c_j^a} A^k(\theta_{a_k}^{k[\cdot]};s)x(s) - \delta_{\theta_{a_k}^{k[\cdot]}}\, x(s), \quad x(t_k) = x_{t_k} \in X, \qquad (4.13)$$

for $s \in [t_k, t_{k+1})$ and $k \in \{1,2,\cdots,\rho\}$.
(4.13) is a linear differential equation with time varying coefficients. We use $\left\{ \tilde{x}^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}(s) \right\}_{s\in[t_k,t_{k+1})}$ to denote the solution path satisfying (4.13). The term $\tilde{x}^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}(t)$ is used interchangeably with $\tilde{x}_t^{\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]}}$.

A remark that will be utilized in subsequent analysis is given below.

Remark 4.2. Let $\tilde{W}^{k(\tau)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t,x)$ denote the value function of the control problem with objective (4.10) and dynamics (2.10) which starts at time τ. One can readily verify that

$$\tilde{W}^{k(\tau)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t,x) = \tilde{W}^{(t_k)}(\theta_{a_k}^{k[(1,a_1)\,(2,a_2)\ldots(k-1,a_{k-1})]};t,x)e^{r(\tau-t_k)}.$$

Climate-preserving technologies will be adopted if the total cooperative gain using these technologies is larger than the expected total cooperative gain using existing technologies, that is $W^{(t_0)}(\theta_0^0;t_0,x_0) > \tilde{W}^{(t_0)}(\theta_0^0;t_0,x_0)$. Moreover, if along the optimal path $\{x^*(s)\}_{s=t_0}^T$, $W^{(t_k)}(\theta_0^0;t_k,x_{t_k}^*) > \tilde{W}^{(t_k)}(\theta_0^0;t_k,x_{t_k}^*)$, for all $k \in \{0,1,2,\cdots,\rho\}$, climate-preserving technologies will be adopted throughout the duration $[t_0,T]$. We first consider the case when climate-preserving technologies are chosen throughout the cooperation duration. In Section 6 the case of partial adoption of climate-preserving technologies will be examined.

4.2. Individually Rational and Dynamically Consistent Imputation

An agreed upon optimality principle must be sought to allocate the cooperative payoff. For $\tau \in [t_k, t_{k+1})$, let $\xi^{k(\tau)i}(\theta_0^0;\tau,x_\tau^*)$ denote the solution imputation (payoff under cooperation) over the period $[t_k,t_{k+1})$ to player $i \in N$ as viewed at time τ. In a dynamic framework individual rationality has to be maintained at every instant of time within the cooperative duration $[t_0, T]$ along the cooperative trajectory $\{x^*(s)\}_{s=t_0}^{T}$. Individual rationality along the cooperative trajectory requires:

$$\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) \geq V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*),$$

$$\text{for} \quad i \in N, \quad \tau \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{0, 1, 2, \cdots, \rho\}, \tag{4.14}$$

along the optimal trajectory $\{x^*(s)\}_{s=t_0}^{T}$.

Since nations are asymmetric and the number of nations may be large, a reasonable solution optimality principle for gain distribution is to share the gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs. To ensure that the cooperative solution is dynamically consistent, the condition of time consistency has to hold. Time consistency requires the solution optimality principle determined at the outset to remain effective at any instant of time throughout the game along the optimal state trajectory. Since all the participants are guided by the same optimality principle at each instant of time, they do not have incentives to deviate from the previously adopted optimal behavior throughout the game. Thus the optimality principle governing the agreed-upon imputation must be maintained throughout the cooperation period to secure time-consistency.

Hence the solution imputation scheme $\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)$, for $i \in N$ and $k \in \{0, 1, 2, \cdots, \rho\}$, has to satisfy:

Condition 4.1.

$$\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) = \frac{V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)}{\sum_{j=1}^{n} V^{k(\tau)j}(\theta_0^0; \tau, x_\tau^*)} W^{(\tau)}(\theta_0^0; \tau, x_\tau^*), \tag{4.15}$$

for $i \in N$ and $\tau \in [t_k, t_{k+1})$ and $k \in \{0, 1, 2, \cdots, \rho\}$, along the optimal path $\{x_\tau^*\}_{\tau=t_0}^{T}$. □

One can easily verify that the imputation scheme in Condition 4.1 satisfies individual rationality. Crucial to the analysis is the formulation of a payment distribution mechanism that would lead to the realization of Condition 4.1. This will be done in the next Section.

5. Payment Distribution Mechanism

To formulate a payment distribution scheme over time so that the agreed upon imputation (4.15) can be realized for any time instant $\tau \in [t_0, T]$ we apply the techniques developed by Yeung and Petrosyan (2004 and 2006b). Let the vector $B^{\theta_0^0}(s, x_s^*) = [B_1^{\theta_0^0}(s, x_s^*), B_2^{\theta_0^0}(s, x_s^*), \cdots, B_n^{\theta_0^0}(s, x_s^*)]$ denote the instantaneous payment to the n nations at time instant s when the state is x_s^* for $s \in [t_0, T]$. A terminal value of $g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x_T^*]$ will be offered to nation i at time T.

To satisfy Condition 4.1 it is required that

$$\xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) = \frac{V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)}{\sum\limits_{j=1}^{n} V^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*)} W^{(\tau)}(\theta_0^0; \tau, x_\tau^*)$$

$$= \int_{\tau}^{T} B_i^{\theta_0^0}(s, x^*(s)) e^{-r(s-\tau)} ds + g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x_T^*] e^{-r(T-\tau)},$$

$$\text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_k, t_{k+1}) \quad \text{and} \quad k \in \{0, 1, 2, \cdots, \rho\}. \tag{5.1}$$

To facilitate further exposition, we use the term $\xi^{k(\tau)i}(\theta_0^0; t, x_t^*)$ which equals

$$\int_{t}^{T} B_i^{\theta_0^0}(s, x^*(s)) e^{-r(s-\tau)} ds + g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x_T^*] e^{-r(T-\tau)}$$

$$= \frac{V^{k(\tau)i}(\theta_0^0; t, x_t^*)}{\sum_{j=1}^{n} V^{k(\tau)i}(\theta_0^0; t, x_t^*)} W^{(\tau)}(\theta_0^0; t, x_t^*)$$

$$= \frac{V^{k(t)i}(\theta_0^0; t, x_t^*)}{\sum_{j=1}^{n} V^{k(t)i}(\theta_0^0; t, x_t^*)} W^{(t)}(\theta_0^0; t, x_t^*) e^{-r(t-\tau)} = \xi^{k(t)i}(\theta_0^0; t, x_t^*) e^{-r(t-\tau)},$$

$$\text{for} \quad i \in N \quad \text{and} \quad \tau \in [t_k, t_{k+1}) \quad \text{and} \quad t \in [\tau, t_{k+1}), \tag{5.2}$$

to denote the present value (with initial time set at τ) of nation i's cooperative payoff over the time interval $[t, T]$.

Theorem 5.1. *A distribution scheme with a terminal payment $g^i_{\theta_0^0}[\bar{x}^i_{\theta_0^0} - x_T^*]$ at time T and an instantaneous payment at time $\tau \in [t_k, t_{k+1})$ equaling*

$$B_i^{\theta_0^0}(\tau, x_\tau^*) = -\left[\xi_t^{k(\tau)i}(\theta_0^0; t, x_t^*) \Big|_{t=\tau} \right]$$

$$- \left[\xi_{x_t^*}^{k(\tau)i}(\theta_0^0; t, x_t^*) \Big|_{t=\tau} \right] \left[\sum_{j=1}^{n} a_j^{\theta_0^0} [\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j \, \psi_h^{\theta_0^0}(\tau, x_\tau)] \right.$$

$$\left. + \sum_{j=1}^{n} b_j^{\theta_0^0} \varpi_j^{\theta_0^0}(\tau, x_\tau)(x_\tau^*)^{1/2} - \delta_{\theta_0^0} \, x_\tau^* \right],$$

$$\textit{for} \quad i \in N \quad \textit{and} \quad \tau \in [t_k, t_{k+1}) \quad \textit{and} \quad k \in \{0, 1, 2, \cdots, \rho\}, \tag{5.3}$$

yield Condition 4.1.

Proof. Since $\xi^{k(\tau)i}(\theta_0^0; t, x_t^*)$ is continuously differentiable in t and x_t^*, using (5.2) and Remarks 3.1 and 4.1 one can obtain:

$$\int_{\tau}^{\tau+\Delta t} B_i^{\theta_0^0}(s, x^*(s)) e^{-r(s-\tau)} ds$$

$$= \xi^{k(\tau)i}(\theta_0^0; \tau, x_\tau^*) - e^{-r\Delta t} \xi^{k(\tau+\Delta t)i}(\theta_0^0; \tau + \Delta t, x_{\tau+\Delta t}^*)$$

$$= \xi^{k(\tau)i}(\theta_0^0;\tau,x_\tau^*) - \xi^{k(\tau)i}(\theta_0^0;\tau+\Delta t,x_{\tau+\Delta t}^*), \tag{5.4}$$

for $i \in N$ and$\tau \in [t_k, t_{k+1})$ and $(\tau+\Delta t) \in [t_k, t_{k+1})$,
where

$$\Delta x_\tau = \left[\ \sum_{j=1}^{n} a_j^{\theta_0^0}[\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j\ \psi_h^{\theta_0^0}(\tau,x_\tau)] + \sum_{j=1}^{n} b_j^{\theta_0^0}\varpi_j^{\theta_0^0}(\tau,x_\tau)(x_\tau^*)^{1/2} - \delta_{\theta_0^0}\, x_\tau^*\ \right]\Delta t$$

$$+o(\Delta t),$$

where $o(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$.
With $\Delta t \to 0$, condition (5.4) can be expressed as:

$$B_i^{\theta_0^0}(\tau,x_\tau^*)\Delta t + o(\Delta t) = -\left[\xi_t^{k(\tau)i}(\theta_0^0;t,x_t^*)\Big|_{t=\tau}\right]\Delta t$$

$$-\left[\xi_{x_t^*}^{k(\tau)i}(\theta_0^0;t,x_t^*)\Big|_{t=\tau}\right]\left[\ \sum_{j=1}^{n} a_j^{\theta_0^0}[\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j\ \psi_h^{\theta_0^0}(\tau,x_\tau^*)]\right.$$

$$\left. + \sum_{j=1}^{n} b_j^{\theta_0^0}\varpi_j^{\theta_0^0}(\tau,x_\tau^*)(x_\tau^*)^{1/2} - \delta_{\theta_0^0}\, x_\tau^*\ \right]\Delta t, \tag{5.5}$$

Dividing (5.6) throughout by Δt, with $\Delta t \to 0$, yields (5.3). Hence Theorem 5.1 follows. □

Theorem 5.1 provides a payoff distribution procedure leading to the satisfaction of Condition 4.1 and hence a dynamically consistent solution will be obtained. When all nations are adopting the cooperative strategies the rate of instantaneous payment that nation $\kappa \in N$ will realize at time τ with the state being x_τ^* can be expressed as:

$$\Re_\kappa^{\theta_0^0}(\tau,x_\tau^*) =$$

$$\left(\alpha^\kappa - \sum_{j=1}^{n} \beta_j^\kappa\{\hat{\alpha}^j + \sum_{h=1}^{n} \hat{\beta}_h^j\ [\hat{\hat{\alpha}}_{\theta_0^0}^h + \hat{\hat{\beta}}_{\theta_0^0}^h\ A_{\theta_0^0}^*(\tau)]\}\right)\{\hat{\alpha}^\kappa + \sum_{h=1}^{n} \hat{\beta}_h^\kappa\ [\hat{\hat{\alpha}}_{\theta_0^0}^h + \hat{\hat{\beta}}_{\theta_0^0}^h\ A_{\theta_0^0}^*(\tau)]\}$$

$$-\hat{c}_\kappa\{\hat{\alpha}^\kappa + \sum_{j=1}^{n} \hat{\beta}_j^\kappa\ [\hat{\hat{\alpha}}_{\theta_0^0}^j + \hat{\hat{\beta}}_{\theta_0^0}^j\ A_{\theta_0^0}^*(\tau)]\} - \varepsilon_\kappa^{\theta_0^0} - c_\kappa^a\left(\frac{b_\kappa^{\theta_0^0}}{2c_\kappa^a}A_{\theta_0^0}^*(\tau)\right)^2 x_\tau^* - h_\kappa^{\theta_0^0}x_\tau^*. \tag{5.6}$$

Since according to Theorem 5.1 under the cooperative scheme an instantaneous payment to nation κ equaling $B_\kappa^{\theta_0^0}(\tau,x_\tau^*)$ at time τ with the state being x_τ^*, a side payment of the value $B_\kappa^{\theta_0^0}(\tau,x_\tau^*) - \Re_\kappa^{\theta_0^0}(\tau,x_\tau^*)$ will be offered to nation κ.

6. Partial Adoption of Climate-preserving Technologies

In this section, we examine the case where climate-preserving technologies are not adopted throughout the duration $[t_0, T]$. Consider the situation when along the optimal path $\{x^*(s)\}_{s=t_0}^{T}$, $W^{(t_k)}(\theta_0^0; t_k, x_{t_k}^*) > \tilde{W}^{(t_k)}(\theta_0^0; t_k, x_{t_k}^*)$, for $k \in \{0, 1, 2, \cdots, \zeta\}$. At time t_ζ, $\tilde{W}^{(t_\zeta)}(\theta_0^0; t_\zeta, x_{t_\zeta}^*) > W^{(t_\zeta)}(\theta_0^0; t_\zeta, x_{t_\zeta}^*)$. This would induce the nations to switch back to non-climate-preserving technologies.

For notational convenience, we denote $\theta_1^{k[\cdot]} = \theta_0^0$, for $k \in \{0, 1, 2, \cdots, \rho\}$ whenever applies. At time t_ζ the climate condition can be expressed as $\theta_1^{\zeta[(1,1)\,(2,1)\ldots(\zeta-1,1)]}$. In the time interval $[t_h, t_{h+1})$, for $h \in \{\zeta+1, \zeta+2, \cdots, \rho\}$, the random variable representing the climate condition can be expressed as $\theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}$.

The imputations $\tilde{\xi}^{h(\tau)i}(\theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta-1,1)\cdots(h-1,a_{h-1})]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta-1,1)\cdots(h-1,a_{h-1})]}})$, for $i \in N$ and $\tau \in [t_h, t_{h+1})$ and $h \in \{\zeta, \zeta+1, \cdots, \rho\}$, which share the gain from cooperation proportional to the nations' relative sizes of expected noncooperative payoffs, require the following condition to hold.

Condition 6.1.

$$\tilde{\xi}^{\zeta(\tau)i}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0}) = \frac{V^{\zeta(\tau)i}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0})}{\sum_{j=1}^{n} V^{\zeta(\tau)j}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0})} \tilde{W}^{\zeta(\tau)}(\theta_0^0; \tau, \tilde{x}_\tau^{\theta_0^0}), \quad \text{for} \quad \tau \in [t_\zeta, t_{\zeta+1}), \tag{6.1}$$

and

$$\tilde{\xi}^{h(\tau)i}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}) = \frac{V^{h(\tau)i}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})}{\sum_{j=1}^{n} V^{h(\tau)j}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})} \tilde{W}^{h(\tau)}(\theta_{a_h}^{h[\cdot]}; \tau, \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}), \tag{6.2}$$

for $\theta_{a_h}^{h[\cdot]} \equiv \theta_{a_h}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}$

$$\in \{\theta_1^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}, \theta_2^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}, \cdots$$

$$\cdots, \theta_{\eta_{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})}}^{h[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}\},$$

$i \in N$ and $\tau \in [t_h, t_{h+1})$, along the optimal path $\left\{ \tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}} \right\}_{\tau=t_h}^{t_{h+1}}$ and $h \in \{\zeta + 1, \zeta + 2, \cdots, \rho\}$.

Following the analysis leading Theorem 5.1, a dynamically consistent distribution scheme can be obtained as:

Theorem 6.1. *A distribution scheme with a terminal payment* $g^i_{\theta_1^{T[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(\rho,a_\rho)]}}[\bar{x}^i_{\theta_1^{T[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(\rho,a_\rho)]}} - x(T)]$ *if* $\theta_{a_T}^{T[(1,1)\,(2,1)\ldots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(\rho,a_\rho)]}$ *occurs at time* T *and an instantaneous payment at time* $\tau \in [t_\zeta, t_{\zeta+1})$ *equaling*

$$\tilde{B}^{\zeta(\tau)i}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0}) = -\left[\tilde{\xi}_t^{\zeta(\tau)i}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})\Big|_{t=\tau}\right]$$

$$-\left[\tilde{\xi}_{\tilde{x}_t}^{\zeta(\tau)i}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})\Big|_{t=\tau}\right]\left[\sum_{j=1}^{n} a_j^{\theta_0^0}[\bar{\alpha}^j + \sum_{h=1}^{n}\bar{\beta}_h^j\,\tilde{\psi}_h^{\zeta}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})]\right.$$

$$\left.+\sum_{j=1}^{n} b_j^{\theta_0^0}\tilde{\omega}_j^{\zeta}(\theta_0^0;\tau,\tilde{x}_\tau^{\theta_0^0})(\tilde{x}_\tau^{\theta_0^0})^{1/2} - \delta_{\theta_0^0}\,\tilde{x}_\tau^{\theta_0^0}\right], \quad \textit{and}$$

at time $\tau \in [t_h, t_{h+1})$ *and* $h \in \{\zeta+1, \zeta+2, \cdots, \rho\}$

$$\tilde{B}^{h(\tau)i}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}) = -\left[\tilde{\xi}_t^{h(\tau)i}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})\Big|_{t=\tau}\right]$$

$$-\left[\tilde{\xi}_{\tilde{x}_t}^{h(\tau)i}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})\Big|_{t=\tau}\right]\left[\sum_{j=1}^{n} a_j^{\theta_{a_h}^{h[\cdot]}}[\bar{\alpha}^j + \sum_{\ell=1}^{n}\bar{\beta}_\ell^j\,\tilde{\psi}_\ell^{h}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})]\right.$$

$$\left.+\sum_{j=1}^{n} b_j^{\theta_{a_h}^{h[\cdot]}}\tilde{\omega}_j^{h}(\theta_{a_h}^{h[\cdot]};\tau,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})(\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}})^{1/2} - \delta_{\theta_{a_h}^{h[\cdot]}}\,\tilde{x}_\tau^{\theta_{a_h}^{h[\cdot]}}\right],$$

$$\textit{for} \quad i \in N, \quad \textit{where} \quad \theta_{a_h}^{h[\cdot]} \equiv \theta_{a_h}^{h[(1,1)\,(2,1)\dots(\zeta,1)(\zeta+1,a_{\zeta+1})\cdots(h-1,a_{h-1})]}, \tag{6.3}$$

yield Condition 6.1.

Proof. Follow the Proof of Theorem 5.1. □

Theorem 6.1 provides a payoff distribution procedure leading to the satisfaction of Condition 6.1 and hence a dynamically consistent solution will be obtained. Though the original climate condition θ_0^0 could not be preserved with partial adoption of climate-preserving technologies, the expected climate deterioration at terminal T is less than that under a noncooperative equilibrium.

7. Concluding Remarks

In this analysis, we present a differential game of pollution management with climate change. Though continual adoption of non-climate-preserving technologies would lead to further irreversible climate deterioration, nations would not to switch to climate-preserving technologies while other nations are using non-climate-preserving technologies. A global ban on non-climate-preserving technologies would unlikely receive unanimous approval because some nations with higher production cost differentials after switching to climate-preserving technologies would become worse off. Through cooperation under which nations would jointly adopt climate-preserving technologies and share the gains in an acceptable scheme could be halted.

In dynamic cooperation, a credible cooperative agreement has to be dynamically consistent. For dynamic consistency to hold the specific optimality principle must remain in effect at any instant of time throughout the game along the optimal state trajectory chosen at the outset. In this paper, dynamically consistent cooperative solutions and analytically tractable payoff distribution procedures are derived.

Moreover, the solutions are obtained in explicit closed-form so one can calculate the intended results with given parametric values. This approach widens the application of cooperative differential game theory to environmental problems where climate change occurs. Since this is the first time cooperative differential games are applied in climate change control, further research along this line is expected.

Appendix 1. Proof of Proposition 3.1.

Using (3.4) and (3.5), system (3.3) can be expressed as:

$$r[A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x + C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)] \; - [\dot{A}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x + \dot{C}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]$$

$$= \Bigg[\; \Bigg(\; \alpha^i - \sum_{j=1}^n \beta_j^i \{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j \; [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k} \; A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} \; \Bigg)$$

$$\Bigg(\; \bar{\alpha}^i + \sum_{h=1}^n \bar{\beta}_h^i \; [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k} \; A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)] \; \Bigg)$$

$$-c_i\{\bar{\alpha}^i + \sum_{j=1}^n \bar{\beta}_j^i \; [\tilde{\alpha}^j_{\theta_{a_\rho}^{\rho[\cdot]}} + \sum_{k=1}^n \tilde{\beta}_k^j \; A^\tau_{\theta_{a_\rho}^{\rho[\cdot]}k}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}$$

$$-c_i^a[\frac{\theta_{a_\rho}^{\rho[\cdot]}}{2c_i^a} A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]^2 x - h_i^{\theta_{a_\rho}^{\rho[\cdot]}} x - \varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}} \; \Bigg]$$

$$+A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \Bigg[\; \textstyle\sum_{j=1}^n a_j^{\theta_{a_\rho}^{\rho[\cdot]}} \{\bar{\alpha}^j + \sum_{h=1}^n \bar{\beta}_h^j \; [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}$$

$$+ \textstyle\sum_{k=1}^n \tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k} \; A_k^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]\} + \sum_{j=1}^n b_j^{\theta_{a_\rho}^{\rho[\cdot]}} \frac{b_j^{\theta_{a_\rho}^{\rho[\cdot]}}}{2c_j^a} A_j^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)x - \delta_{\theta_{a_\rho}^{\rho[\cdot]}} x \; \Bigg], \quad \text{and} \tag{A.1}$$

$$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T)x + C_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) \; = \sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}$$

$$\times g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} [\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}} - x(T)] e^{-r(T-t_\rho)};$$

for $i \in N$. (A.2)

For (A.1) and (A.2) to hold, it is required that

$$\dot{A}_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) = (r + \delta_{\theta_{a_\tau}^{\tau[\cdot]}}) A_i^\tau(\theta_{a_\tau}^{\tau[\cdot]};t) - A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t) \sum_{\substack{j=1 \\ j \neq i}}^n \frac{(b_j^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{2c_j^a} A_j^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)$$

$$-\frac{(b_i^{\theta_{a_\rho}^{\rho[\cdot]}})^2}{4c_i^a}[A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};t)]^2 + h_i^{\theta_{a_\rho}^{\rho[\cdot]}}, \tag{A.3}$$

$$A_i^\rho(\theta_{a_\rho}^{\rho[\cdot]};T) = -\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}} \lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]} g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}; \tag{A.4}$$

and $\dot{C}_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)$=

$$rC_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)-\left(\ \alpha^i-\sum_{j=1}^{n}\beta_j^i\{\bar{\alpha}^j+\sum_{h=1}^{n}\bar{\beta}_h^j\ [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\ A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}\ \right)$$

$$\left(\ \bar{\alpha}^i+\sum_{h=1}\bar{\beta}_h^i\ [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\ A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\ \right)$$

$$+c_i\{\bar{\alpha}^i-\sum_{j=1}^{n}\bar{\beta}_j^i\ [\tilde{\alpha}^j_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^j_{\theta_{a_\rho}^{\rho[\cdot]}k}\ A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}+\varepsilon_i^{\theta_{a_\rho}^{\rho[\cdot]}}$$

$$-A_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)\left[\ \sum_{j=1}^{n}a_j^{\theta_{a_\rho}^{\rho[\cdot]}}\{\bar{\alpha}^j+\sum_{h=1}^{n}\bar{\beta}_h^j\ [\tilde{\alpha}^h_{\theta_{a_\rho}^{\rho[\cdot]}}+\sum_{k=1}^{n}\tilde{\beta}^h_{\theta_{a_\rho}^{\rho[\cdot]}k}\ A_k^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};t)]\}\ \right],\tag{A.5}$$

$$C_i^{\rho}(\theta_{a_\rho}^{\rho[\cdot]};T)=\sum_{a_T=1}^{\eta_{T[(1,a_1)(2,a_2)\dots(\rho,a_\rho)]}}\lambda_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}$$
$$\times g^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}\bar{x}^i_{\theta_{a_T}^{T[(1,a_1)\,(2,a_2)\dots(\rho,a_\rho)]}}.\tag{A.6}$$

Hence Proposition 3.1 follows. *Q.E.D.*

Appendix 2. Proof of Proposition 4.1.

Substituting (4.4) and (4.6) into (4.3) and using (4.7) one obtains:

$$r[A^*_{\theta_0^0}(t)x+C^*_{\theta_0^0}(t)]-[\dot{A}^*_{\theta_0^0}(t)x+\dot{C}^*_{\theta_0^0}(t)]\ =$$

$$\sum_{\kappa=1}^{n}\Bigg[\ \left(\alpha^{\kappa}-\sum_{j=1}^{n}\beta_j^{\kappa}\{\hat{\alpha}^j+\sum_{h=1}^{n}\hat{\beta}_h^j\ [\hat{\hat{\alpha}}^h_{\theta_0^0}+\hat{\hat{\beta}}^h_{\theta_0^0}\ A^*_{\theta_0^0}(t)]\}\right)$$

$$\times\{\hat{\alpha}^{\kappa}+\sum_{h=1}^{n}\hat{\beta}_h^{\kappa}\ [\hat{\hat{\alpha}}^h_{\theta_0^0}+\hat{\hat{\beta}}^h_{\theta_0^0}\ A^*_{\theta_0^0}(t)]\}$$

$$-\hat{c}_{\kappa}\{\hat{\alpha}^{\kappa}+\sum_{j=1}^{n}\hat{\beta}_j^{\kappa}\ [\hat{\hat{\alpha}}^j_{\theta_0^0}+\hat{\hat{\beta}}^j_{\theta_0^0}\ A^*_{\theta_0^0}(t)]\}-c_{\kappa}^a[\frac{b_{\kappa}^{\theta_0^0}}{2c_{\kappa}^a}A^*_{\theta_0^0}(t)]^2x-\varepsilon_{\kappa}^{\theta_0^0}-h_{\kappa}^{\theta_0^0}x\ \Bigg]$$

$$+A^*_{\theta_0^0}(t)\left[\ \sum_{j=1}^{n}a_j^{\theta_0^0}\{\hat{\alpha}^j+\sum_{h=1}^{n}\hat{\beta}_h^j\ [\hat{\hat{\alpha}}^h_{\theta_0^0}+\hat{\hat{\beta}}^h_{\theta_0^0}\ A^*_{\theta_0^0}(t)]\}+\sum_{j=1}^{n}\frac{(b_j^{\theta_0^0})^2}{2c_j^a}A^*_{\theta_0^0}(t)x-\delta_{\theta_0^0}x\ \right],\tag{A.7}$$

$$[A^*_{\theta_0^0}(T)x+C^*_{\theta_0^0}(T)]=\sum_{\kappa=1}^{n}g^{\kappa}_{\theta_0^0}[\bar{x}^{\kappa}_{\theta_0^0}-x(T)].\tag{A.8}$$

For (A.7) and (A.8) to hold, it is required that

$$\dot{A}^*_{\theta_0^0}(t)=(r+\delta_{\theta_0^0})\ A^*_{\theta_0^0}(t)-\sum_{j=1}^{n}\frac{(b_j^{\theta_0^0})^2}{2c_j^a}[A^*_{\theta_0^0}(t)]^2+\sum_{j=1}^{n}h_j^{\theta_0^0},\tag{A.9}$$

$$A^*_{\theta_0^0}(T)=-\sum_{j=1}^{n}g^j_{\theta_0^0};\tag{A.10}$$

$$\dot{C}^*_{\theta^0_0}(t) = rC^*_{\theta^0_0}(t)$$

$$-\sum_{\kappa=1}^{n}\Bigg[\ \Bigg(\alpha^\kappa - \sum_{j=1}^{n}\beta_j^\kappa\{\hat{\alpha}^j + \sum_{h=1}^{n}\hat{\beta}_h^j\ [\hat{\alpha}^h_{\theta^0_0} + \hat{\hat{\beta}}^h_{\theta^0_0}\ A^*_{\theta^0_0}(t)]\}\Bigg)$$

$$\times\{\hat{\alpha}^\kappa + \sum_{h=1}^{n}\hat{\beta}_h^\kappa\ [\hat{\alpha}^h_{\theta^0_0} + \hat{\hat{\beta}}^h_{\theta^0_0}\ A^*_{\theta^0_0}(t)]\} - c_\kappa\{\hat{\alpha}^\kappa + \sum_{j=1}^{n}\hat{\beta}_j^\kappa\ [\hat{\alpha}^j_{\theta^0_0} + \hat{\hat{\beta}}^j_{\theta^0_0}\ A^*_{\theta^0_0}(t)]\} - \varepsilon_\kappa^{\theta^0_0}\ \Bigg]$$

$$-A^*_{\theta^0_0}(t)\Bigg[\ \sum_{j=1}^{n} a_j^{\theta^0_0}\{\hat{\alpha}^j + \sum_{h=1}^{n}\hat{\beta}_h^j\ [\hat{\alpha}^h_{\theta^0_0} + \hat{\hat{\beta}}^h_{\theta^0_0}\ A^*_{\theta^0_0}(t)]\}\ \Bigg], \tag{A.11}$$

$$C^*_{\theta^0_0}(T) = \sum_{j=1}^{n} g^j_{\theta^0_0}\bar{x}^j_{\theta^0_0}. \tag{A.12}$$

Hence Proposition 4.1 follows. *Q.E.D.*

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On Polytope of (0-1)-normal Big Boss Games: Redundancy and Extreme Points

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Abstract The system of non redundant constraints for polytope of monotonic (0-1)-normal big boss games is obtained. The explicit representation of some types of extreme points of this polytope as well as the corresponding Shapley and consensus values formulas are given. We provide the characterization of extreme elements of set of such monotonic (0-1)-normal big boss games that all weak players are symmetric.

Keywords: cooperative game, big boss game, (0-1)-normal form, extreme points, Shapley value, consensus value.

1. Introduction

The class of big boss games was introduced to model economic, social and political situations in which one of the participants has a greater possibilities (power) than others (see, for example (Hubert and Ikonnikova, 2011)), (Tijs, et al., 2005) (O'Neill, 1982), (Tijs, 1990), (Aumann and Maschler, 1985), (Branzei, et al., 2006)). In (Muto, et al., 1988) the big boss games as well as strong big boss games were determined by means of three conditions: monotonicity, boss property and union property. Later appeared the work (Tijs, 1990) in which monotonicity condition was replaced by nonnegativity of characteristic function and marginal vector. The general (Branzei and Tijs, 2001) and total (Muto, et al., 1988) big boss games were also introduced. All types of big boss games are extensively studied. Moreover, the results received for clan games (Potters et al., 1989) are applicable to big boss games because the cone of each type of big boss games is a subset of cone of corresponding clan games. One of cooperative game theory problems is the characterization of extreme directions of polyhedral cones of various classes of games and description the behavior of solution concepts defined on these cones (Tijs and Branzei, 2005). The extreme directions of cone of non-monotonic clan games were described in (Potters et al., 1989). If the clan consists of one player these and only these directions define the cone of non-monotonic big boss games. To our knowledge the extreme elements of set of monotonic big boss games are not yet characterized.

Since big boss games can be converted to (0-1)-normal form without changing their essential structure and the most solution concepts satisfy on this class games the relative invariance with respect to strategic equivalence, this paper focus on (0-1)-normalized big boss games. At normalization the cone of monotonic big boss games will be transformed to $(2^{n-1}-2)$-dimensional polytope $\mathbf{P}^n$ which can be described by its extreme points. From Theorem 4.1 in (Potters et al., 1989) it follows that only simple games are the extreme points of polytope of nonmonotonic (0-1)-normal big boss games. But for $\mathbf{P}^n$ this is hot true.

The paper has the following contents. Next section recall the facts of cooperative game theory which are useful later. The system of non-redundant constraints for $\mathbf{P}^n$ is described in Section 3. Section 4 is denoted to extreme points of $\mathbf{P}^n$ and their Shapley and consensus values. The characterization of extreme elements of set of monotonic (0-1)-normal big boss games with symmetric weak players is given in last section.

2. Preliminaries

A *cooperative TU-game* is a pair (N, ν) where $N = \{1, 2, \ldots, n\}$ is a *player set* and $\nu \in G^N = \{g : 2^N \to \mathbf{R} \mid g(\oslash) = 0\}$ is a set function. Often ν and (N, ν) will be identified. A subset of N is called a *coalition* and $\nu(S)$ is the *worth* of coalition S. A vector $x \in \mathbf{R}^n$ is called an *allocation*. For any $S \in 2^N$ and $x \in \mathbf{R}^n$ let $x(S) = \sum_{i \in S} x_i$ and $x(\emptyset) = 0$. Two players $i, j \in N$ are *symmetric* in (N, ν) if $\nu(S \cup i) = \nu(S \cup j)$ for every $S \subseteq N \setminus \{i, j\}$. We say that players of coalition S with $|S| \geq 2$ are symmetric in (N, ν) if each pair of players of the coalition is symmetric in (N, ν). Denote by $M(\nu) \in \mathbf{R}^n$ the *marginal vector* (marginal) of game $\nu \in G^N$, i.e. $M_i(\nu) = \nu(N) - \nu(N \setminus i)$, $i \in N$. A TU-game is called:
- *monotonic* if $\nu(T) \leq \nu(H)$ for all $T \subset H \subseteq N$,
- *simple* if $\nu(S) \in \{0, 1\}$ for all $S \subseteq N$ and $\nu(N) = 1$,
- *(0-1)-normal* if $\nu(N) = 1$ and $\nu(i) = 0$ for all $i \in N$,
- *essential* if $\sum_{i \in N} \nu(i) < \nu(N)$,
- *clan game* with nonempty coalition $CLAN$ as clan (Potters et al., 1989) if:

$\nu \geq 0$ and $M(\nu) \geq 0$,
$\nu(S) = 0$ if $CLAN \not\subset S$,
$\nu(N) - \nu(S) \geq \sum_{i \in N \setminus S} M_i(\nu)$ if $CLAN \subset S$.

Later we need formulas for the *Shapey value* Sh (Shapley, 1953), the *equal surplus division solution* E and the *consensus value* K (Ju, et al., 2006). These values are given by

$$Sh_i(\nu) = \sum_{S: i \notin S} \rho_S(\nu(S \cup i) - \nu(S)),\ \rho_S = \frac{|S|!(n - |S| - 1)!}{n!},$$

$$E_i(\nu) = \nu(i) + \frac{\nu(N) - \sum_{j \in N} \nu(j)}{n},\ i \in N, \quad K(\nu) = \frac{E(\nu) + Sh(\nu)}{2}.$$

For simple game the Shapley value formula boils down to

$$Sh_i(\nu) = \sum_{S \subseteq \mathcal{R}_i} \rho_S,\ i \in N,$$

where $\mathcal{R}_i = \{S \subseteq N \setminus i : \nu(S) = 0,\ \nu(S \cup i) = 1\}$. For (0-1)-normal game the consensus value is determined by

$$K_i(\nu) = \frac{1}{2n} + \frac{Sh_i(\nu)}{2},\ i \in N,$$

because $E_i(\nu) = \frac{1}{n},\ i \in N$.

For any set $G \subseteq G^N$ a *value* on G is a function $\phi : G \to \mathbf{R}^n$ which assigns to every $\nu \in G$ a vector $\phi(\nu)$, where $\phi_i(\nu)$ represents the payoff to player i in ν. We shall use two axioms to be satisfied by $\phi(\nu)$.

Efficiency: $\sum_{i\in N}\phi_i(\nu)=\nu(N)$ for all $\nu\in G$.

Symmetry: for all $\nu\in G$ and every symmetric players $i,j\in N$, $\phi_i(\nu)=\phi_j(\nu)$. Known that the Shapley value and the consensus value satisfy this axioms. The *core* (Gillies, 1953) of game $\nu\in G^N$ is a bounded polyhedral set (polytope) $C(\nu)=\{x\in R^n : x(N)=\nu(N),\ x(S)\geq\nu(S),\ S\subset N\}$. The sets of integer and non-integer extreme points of polytope $\mathbf{P}$ will be denoted by $ext_I(\mathbf{P})$ and $ext_{NI}(\mathbf{P})$ respectively. The cardinality of set S is written as $|S|$. The rank of matrix A is denoted as $rank(A)$.

3. Minimal test for big boss game

A game $\nu\in G^N$ $(n\geq 3)$ is called a *big boss game* with player 1 as big boss (Muto, et al., 1988) if:
(a) ν is monotonic,
(b) $\nu(T)=0$ for all $T\subset N$ with $1\notin T$ (*boss property)*,
(c) $\nu(N)-\nu(T)\geq\sum_{i\in N\setminus T}M_i(\nu)$ for all $T\subseteq N$ with $1\in T$ (*union property*).

Inessential games are not interesting. Any essential game has the unique (0-1)-normal form. Denote by $\mathbf{P}^n$ the polytope of all monotonic (0-1)-normal big boss games with player 1 as big boss. This set is determined by

$$\nu(N)=1,\ \nu(T)=0\ \ whenever\ 1\notin T\ ore\ T=\{1\}, \tag{1}$$

$$\nu(H)\geq\nu(T),\ T\subset H\subseteq N, \tag{2}$$

$$-\nu(T)+\sum_{i\in N\setminus T}\nu(N\setminus i)\geq n-|T|-1,\quad T\ni 1,\ T\subseteq N. \tag{3}$$

The following lemma shows that (1)-(3) is equivalent to a restricted system.

Lemma 1. *Let $\nu\in G^N$. Then $\nu\in\mathbf{P}^n$ iff it satisfies (1) and conditions*

$$\nu(H)\geq\nu(T),\quad T\ni 1,\ T\subset H\subseteq N,\ 1\leq|T|=|H|-1,\ |H|\neq n-1, \tag{4}$$

$$-\nu(T)+\sum_{i\in N\setminus T}\nu(N\setminus i)\geq n-|T|-1,\quad T\ni 1,\quad |T|\leq n-2,\quad T\subset N. \tag{5}$$

Proof. If $\nu\in\mathbf{P}^n$ then it satisfies (1),(4),(5) because the system (2)-(3) contains all inequalities in (4)-(5). Now take $\bar\nu$ satisfying (1),(4),(5). The constraints in (3) corresponding to $T=N$ and $T=N\setminus i$, $i\in N\setminus 1$, are trivial. So, the systems (3) and (5) are equivalent. Obviously $\bar\nu$ satisfies (2) for coalitions $T=\emptyset$ and $H=\{i\}$, $i\in N$. Hence, it suffices to show that $\bar\nu$ satisfies (2) for following pairs of T and H.

1. $T\ni 1$, $T\subset H\subset N$, $|H|=n-1$, $|T|=n-2$. The inequalities in (2) corresponding to such coalitions are the form

$$\nu(N\setminus k)\geq\nu(N\setminus\{k,e\},\ k\in N\setminus 1,\ e\in N\setminus\{1,k\}. \tag{6}$$

From (5) follows $-\bar\nu(N\setminus\{k,e\})+\bar\nu(N\setminus k)+\bar\nu(N\setminus e)\geq 1$. Due to (1) and (4) it holds that $1\geq\bar\nu(N\setminus e)$. The summing of two last inequalities gives that $\bar\nu$ satisfies (6).

2. $T\ni 1$, $T\subset H\subseteq N$, $|H|>|T|+1$. The corresponding inequalities in (2) are satisfied for $\bar\nu$ since the binary relation "$\geq$" is transitive.

3. $T \not\ni 1$, $T \subset H \subset N$. From case 1 we know that $\bar{\nu}$ satisfies the inequalities in (2) for such T and H that $T \ni 1$, $T \subset H \subseteq N$, $|T| = |H| - 1$. Together with (1) this implies that $\bar{\nu}(S) \geq 0$ for all $S \subseteq N$. Because $\bar{\nu}(S) = 0$ for all $S \not\ni \{1\}$, we obtain that $\bar{\nu}$ satisfies (2). □

A constraint in a linear system is called *redundant* if the removal of this constraint from the system does not affect the feasible region. Next theorem provides the system of non-redundant conditions for $\mathbf{P}^n$.

Theorem 1. *The system (1),(4),(5) is non-redundant.*

Proof. Let $\hat{\nu} \in G^N$ be given by

$$\hat{\nu}(S) = \begin{cases} \frac{|S|-1}{n}, & S \ni 1,\ |S| \leq n-2, \\ \frac{|S|}{n}, & S \ni 1,\ |S| \geq n-1, \\ 0, & otherwise, \end{cases}$$

for all $S \subseteq N$. Obviously $\hat{\nu} \in \mathbf{P}^n$ and $\hat{\nu}$ is the interior point of system (4),(5) feasible region. None of constraints in the system (1) is implied by the others because they are linear independent. The table 1 contains such games $\nu \in G^N$ that satisfy (1), (4), (5) except the unique constraint (corresponding to coalitions given in the first column). Vectors ν^1-ν^3 do not satisfy one of inequalities in (4) and for ν^4-ν^5 one of inequalities in (5) is violated. It is assumed that $T \subset H \subseteq N$, $|T| = |H| - 1$ and $S \subseteq N$. □

Corollary 3.1. *The polytope $\mathbf{P}^n$ is $(2^{n-1} - 2)$-dimensional.*

Proof. The number of constraints in (1) is $2^{n-1} + 2$. Using the fact that $\mathbf{P}^n \subset \mathbf{R}^{2^n}$ and non-redundancy the system (1),(4),(5) we obtain $dim(\mathbf{P}^n) = 2^{n-1} - 2$. □

Corollary 3.2. $\mathbf{P}^3$ *and* $\mathbf{P}^4$ *are the integral polytopes.*

Proof. Every $\nu \in \mathbf{P}^n$ is (0-1)-normal monotonic clan game with $CLAN = 1$. But in cases P^3 and $\mathbf{P}^4$ the monotonicity conditions (4) are transformed in bounds on variables: $\nu(1,i) \geq 0$, $\nu(N \setminus i) \leq 1$, $i \in N \setminus 1$. Theorem 4.1 in (Potters et al., 1989) implies that $\mathbf{P}^3$ and $\mathbf{P}^4$ have only integer extreme points. □

We have calculated all extreme points of $\mathbf{P}^5$ and partitioned the set $ext_{NI}(\mathbf{P}^5)$ into seven equivalence classes. The representatives of these classes are given in Table 2. Each class contains such games that differ only the numbers of players from $N \setminus 1$. Note that (0-1)-normal form of 5-person game given in counterexample 4 in (Potters et al., 1989) coincides with $\Psi^{-1}(\bar{\nu}^1)$.

4. Extreme points of polytope $\mathbf{P}^n$

Since $\mathbf{P}^n$ is contained in the unit hypercube, the simple games belonging to $\mathbf{P}^n$ are its integer extreme points. To make the following analysis simple, consider the polytope $\tilde{\mathbf{P}}^n$ determined by

$$\nu(T) \geq 0 \ if \ T \in \Omega \ and \ |T| = 2, \quad \nu(T) \leq 1 \ if \ T \in \Omega \ and \ |T| = n-1, \tag{7}$$

$$\nu(H) \geq \nu(T) \ if \ T, H \in \Omega,\ T \subset H,\ 2 \leq |T| = |H| - 1,\ |H| \leq n-2, \tag{8}$$

Table1: Non big boss games

Fixed coalitions	Games
$T \ni 1,\ \lvert H\rvert \le n-2,$	$\nu^1(S) = \begin{cases} 1, & (T \subseteq S) \wedge (S \neq H) \vee (\lvert S\rvert = n-1) \wedge (S \neq N \setminus 1), \\ 0, & otherwise. \end{cases}$
$T \ni 1,\ H = N,$	$\nu^2(S) = \begin{cases} 1, & S \neq T,\ \lvert S\rvert = n-1,\ S \neq N \setminus 1 \\ 2, & S = T, \\ 0, & otherwise. \end{cases}$
$T = 1,\ \lvert H\rvert = 2,$	$\nu^3(S) = \begin{cases} 1, & S = n-1,\ S \neq N \setminus 1 \\ -1, & S = H, \\ 0, & otherwise. \end{cases}$
$T \ni 1,\ 2 \le \lvert T\rvert \le n-2,$	$\nu^4(S) = \begin{cases} \frac{\lvert S\rvert}{n-1}, & S = T, \\ 0, & S \not\ni 1, \\ \frac{\lvert S\rvert - 1}{n-1}, & otherwise. \end{cases}$
$T = \{1\},$	$\nu^5(S) = \begin{cases} 1, & \lvert S\rvert = n, \\ \frac{n-3}{n-2}, & S \ni 1,\ \lvert S\rvert = n-1, \\ 0, & otherwise. \end{cases}$

Table2: Types of non integer extreme points of $\bar{P}^5$

	{1,2}	{1,3}	{1,4}	{1,5}	{1,2,3}	{1,2,4}	{1,2,5}	{1,3,4}	{1,3,5}	{1,4,5}	$N\setminus5$	$N\setminus4$	$N\setminus3$	$N\setminus2$
$\bar{\nu}^1$	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	$\frac{1}{2}$	$\frac{1}{2}$
$\bar{\nu}^2$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\bar{\nu}^3$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
$\bar{\nu}^4$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\bar{\nu}^5$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\bar{\nu}^6$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{4}{4}$
$\bar{\nu}^7$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}$

$$-\nu(T) + \sum_{i \in N \setminus T} \nu(N \setminus i) \geq n - |T| - 1), \quad T \in \Omega, \quad |T| \leq n - 3, \tag{9}$$

$$\sum_{i \in N \setminus 1} \nu(N \setminus i) \geq n - 2, \tag{10}$$

where $\Omega = \{S \in 2^N : 2 \leq |S| \leq n-1, \ S \ni 1\}$. The system (7)-(10) is obtained from (1),(4),(5) by elimination the variables which have constant value over $\mathbf{P}^n$. Moreover, the monotonicity condition (4) are decomposed into three parts (in order to select upper and lower bounds on variables). The inequality corresponding to coalition $T = \{1\}$ was selected from system (5). Thus, the polytope $\bar{\mathbf{P}^n}$ is contained in the $(2^{n-1}-2)$-dimensional Euclidean space whose coordinates refer to the coalitions $S \in \Omega$. Theorem 1 and Corollary 1 imply that $dim(\bar{\mathbf{P}^n}) = dim(\mathbf{P}^n)$, i.e. the polytope $\bar{\mathbf{P}^n}$ is full-dimensional. So, the system (7)-(10) is the unique non-redundant system which specifies $\bar{\mathbf{P}^n}$. The polytopes $\bar{\mathbf{P}^n}$ and $\mathbf{P}^n$ are combinatorially equivalent since there is one-to-one map $\Psi : \mathbf{P}^n \to \bar{\mathbf{P}^n}$ saving the adjacency of faces. For any $\nu \in \mathbf{P}^n$ the vector $\Psi(\nu) = (\nu(\bar{S}))_{S \in \Omega}$ is the restriction of vector $(\nu(S))_{S \in 2^N}$ to those coordinates which correspond to $S \in \Omega$. Conversely, having $\bar{\nu} \in \bar{\mathbf{P}^n}$ we obtain the game $\Psi^{-1}(\bar{\nu}) = \nu \in \mathbf{P}^n$ by adding values $\nu(N)$ and $\nu(S)$, $S \in 2^N \setminus \Omega$, determined by (1). The following theorem describes some elements of $ext_{NI}(\mathbf{P}^n)$.

Theorem 2. *Let $n \geq 5$, $(i_2, ..., i_n)$ be an ordering on $N \setminus 1$, $L = (1, i_2, ..., i_l)$, $2 \leq l \leq n-2$ and for all $S \in \Omega$*

$$\bar{\nu}^0(S) = \begin{cases} \dfrac{n-2}{n-1}, & |S| = n-1, \\ \dfrac{1}{n-1}, & otherwise, \end{cases} \qquad \bar{\nu}^L(S) = \begin{cases} 1, & |S| = n-1, \ L \not\subseteq S, \\ \dfrac{n-|L|-1}{n-|L|}, & |S| = n-1, \ L \subseteq S, \\ 0, & |S| < n-1, \ S \subseteq L, \\ \dfrac{1}{n-|L|}, & otherwise. \end{cases}$$

Then $\nu^0 = \Psi^{-1}(\bar{\nu}^0)$ and $\nu^L = \Psi^{-1}(\bar{\nu}^L)$ are the extreme points of $\mathbf{P}^n$.

Proof. Let us prove that $\bar{\nu}^0 \in ext(\bar{\mathbf{P}^n})$. The system (7)-(10) contains $d = 2^{n-1} - 2$ variables. The subsystem (8) contains only $(d-n+1)$ of them and its matrix is the incidence matrix of connected graph in which the set of vertices equals the set of such coalitions $S \in \Omega$ that $|S| \leq n-2$. The rank of this matrix is $(d-n)$. Choose $(d-n)$ linear independent constraints in (8) and denote by Θ the set of associated pairs of coalitions T and H. The system

$$\nu(H) = \nu(T), \ (T, H) \in \Theta, \tag{11}$$

$$-\nu(T) + \sum_{i \in N \setminus T} \nu(N \setminus i) = n - 3, \ T \in \Omega, \ |T| = 2, \tag{12}$$

$$\sum_{i \in N \setminus 1} \nu(N \setminus i) = n - 2, \tag{13}$$

contains d variable as much as equations. The elimination $(d-n)$ variables from (11) and substitution them in (12)-(13) gives the system $A\nu = b$ where A is square matrix of dimension n, $b = (n-3, ..., n-3, n-2) \in \mathbf{R}^n$. By transposition of columns and rows the matrix A can be represented in the form

$$\begin{pmatrix} -e_{n-1}^T & D \\ 0 & e_{n-1} \end{pmatrix}$$

where $e_{n-1} = (1, ..., 1)$ is the $(n-1)$-dimensional row vector and D is the square matrix of dimension $(n-1)$ with $d_{ij} = 0$ if $i = j$, $d_{ij} = 1$ if $i \neq j$. Vector $\bar{\nu}^0$ is the solution of system (11)-(13) and it is unique because $rank(A) = rank(D) + 1 = n$. It is easy to see that $\bar{\nu}^0 \in \bar{\mathbf{P}}^n$. Hence,

$$\bar{\nu}^0 \in ext(\bar{\mathbf{P}}^n) \implies \nu^0 \in ext(\mathbf{P}^n).$$

Analogously one proves that $\bar{\nu}^L(S) \in ext(\bar{\mathbf{P}}^n)$. □

Propositions 1, 2 (below) provide the explicit Shapley and consensus values representation for some integer and noninteger extreme points of $\mathbf{P}^n$.

Proposition 1. *Let $\bar{\nu}^k$ is determined for all $k \in \{2, \ldots, n-1\}$ and $S \in \Omega$ by*

$$\bar{\nu}^k(S) = \begin{cases} 1, & |S| \geq k, \\ 0, & otherwise. \end{cases}$$

Then $\nu^k = \Psi^{-1}(\bar{\nu}^k) \in ext_I(\mathbf{P}^n)$ and

$$K_1(\nu^k) = \frac{n-k+2}{2n}, \quad Sh_1(\nu^k) = \frac{n-k+1}{n},$$

$$K_i(\nu^k) = \frac{n+k-2}{2n(n-1)}, \quad Sh_i(\nu^k) = \frac{k-1}{n(n-1)}, \quad i \in N \setminus 1.$$

Proof. Fix $k \in \{2, \ldots, n-1\}$. The vector $\bar{\nu}^k$ obviously satisfies (7)-(8). It also satisfies (9)-(10) since $\bar{\nu}^k(N \setminus i) = 1$, $i \in N \setminus 1$. So, $\bar{\nu}^k \in \bar{\mathbf{P}}^n$. This implies that $\nu^k \in \mathbf{P}^n$. Further, $\nu^k \in ext_I(\mathbf{P}^n)$ because it is a simple game. Take $i^* \in N \setminus 1$. Then $\mathcal{R}_{i^*} = \{S \subseteq N \setminus i^* : |S| = k-1\}$ and $\rho_S = \frac{(k-1)!(n-k)!}{n!}$ for all $S \in R_{i^*}$. The substitution ρ_S and R_{i^*} in the Shapley value formula for simple game gives $Sh_{i^*}(\nu^k) = \rho_S|R_{i^*}| = \rho_S \binom{n-2}{k-2} = \frac{k-1}{n(n-1)}$. All weak players are symmetric in ν^k. By *Symmetry* $Sh_j(\nu^k) = Sh_{i^*}(\nu^k)$, $j \in N \setminus \{1, i^*\}$. From *Efficiency* follows $Sh_1(\nu^k) = \nu(N) - \sum_{i \in N \setminus 1} Sh_i(\nu^k) = 1 - \frac{k-1}{n} = \frac{n-k+1}{n}$. The consensus value of game ν^k is defined by formula for (0-1)-normal games. □

Proposition 2. *The Shapley and consensus values for game $\nu^0 \in ext_{NI}(\mathbf{P}^n)$ determined in Theorem 2 are*

$$Sh_1(\nu^0) = \frac{3n-6}{n(n-1)}, \quad K_1(\nu^0) = \frac{4n-7}{2n(n-1)},$$

$$Sh_i(\nu^0) = \frac{n^2-4n+6}{n(n-1)^2}, \quad K_i(\nu^0) = \frac{2n^2-6n+7}{2n(n-1)^2} \quad for \quad i \in N \setminus 1.$$

Proof. Take $i^* \in N \setminus 1$. Then for all $S \subseteq N$

$$\nu^0(S \cup i^*) - \nu^0(S) = \begin{cases} \frac{1}{n-1}, & S = \{1\} \text{ or } S = N \setminus i^*, \\ \frac{n-3}{n-1}, & S \ni 1,\ |S| = n-2,\ S \not\ni i^*, \\ 0, & \text{otherwise.} \end{cases}$$

The number of coalitions satisfying $S \ni 1$, $|S| = n-2$, $S \not\ni i^*$ is $(n-2)$ and $\rho_S = \frac{1}{n(n-1)}$ for such S. Further, $\rho_{\{1\}} = \frac{1}{n(n-1)}$, $\rho_{N\setminus i^*} = \frac{1}{n}$. By using the Shapley value formula we obtain $Sh_{i^*}(\nu^0) = \frac{1}{n(n-1)^2} + \frac{(n-3)(n-2)}{n(n-1)^2} + \frac{1}{n(n-1)} = \frac{n^2-4n+6}{n(n-1)^2}$. By *Symmetry* and *Efficiency* $Sh_1(\nu^0) = 1 - \frac{n^2-5n+7}{n(n-1)} = \frac{3n-6}{n(n-1)}$. The substitution $Sh(\nu^0)$ in consensus value formula gives $K(\nu^0)$. □

The core $C(\nu) = \{x \in \mathbf{R}^n : x(N) = \nu(N),\ 0 \le x_i \le M_i(\nu),\ i \in N \setminus 1\}$ of each game $\nu \in \mathbf{P}^n$ is determined by marginal vector only (Muto, et al., 1988). So, all games in $\mathbf{P}^n$ having identical marginals have the same core. If $C(\nu)$ is a singleton, i.e. $C(\nu) = \{x^c\}$, then the bargaining set (Aumann and Maschler, 1964), kernel (Davis and Maschler, 1965) and lexicore (Funaki, et al., 2007) coincide with x^c. Moreover, any core selector (for example, nucleolus (Schmeidler, 1969), $\tau-$ value (Tijs, 1981), $AL-$value (Tijs, 2005)) coincides with x^c. Thus, x^c should reflect many principles of fairness. However, for games with zero $M_i(\nu)$, $i \in N \setminus 1$, we have $x^c = (1, 0, \ldots, 0)$. According to x^c the entire unit of surplus is allocated to player 1 (boss) that ignores the productive role of other players. Such games are in particular ν^k determined in Proposition 1 and all games in their convex hull and also all games in G^N having corresponding (0-1)-form. At the same time, by formulas from Proposition 1 we obtain different consensus and Shapley values. For example, take two 6-person games ν^2 and ν^5. Then

$$K(\nu^2) = (\frac{1}{2}, \frac{1}{10}, \ldots, \frac{1}{10}), \quad K(\nu^5) = (\frac{1}{4}, \frac{3}{20}, \ldots, \frac{3}{20})$$

,

$$Sh(\nu^2) = (\frac{5}{6}, \frac{1}{30}, \ldots, \frac{1}{30}), \quad Sh(\nu^5) = (\frac{1}{3}, \frac{2}{15}, \ldots, \frac{2}{15})$$

.

Thus, for $\nu \in co(\{\nu^k\}_{k=2}^{n-1})$ the consensus and Shapley values prescribes a rather natural outcomes.

5. *L*-symmetrical big boss games

We name a game $\nu \in \mathbf{P}^n$ *l-symmetric* if each pair of powerless players $i, j \in N \setminus 1$ is symmetric in ν. Denote by $\mathbf{S}P^n$ the class of l-symmetric games $\nu \in \mathbf{P}^n$. Let $\mathbf{X}$ be the set of all (0,1)-vectors $x = (x_2, \ldots, x_{n-2})$, $s = |S|$, $J = \{2, ..., n-2\}$ and $\bar{\nu} = \Psi(\nu)$ whenever $\nu \in \mathbf{SP}^n$. Next theorem characterizes extreme points of $\mathbf{SP}^n$. It shows also that all non-integer extreme points belongs to $(2^n - n - 1)$-dimensional face

$$\{\nu \in \mathbf{SP}^n : \nu(S) = \frac{n-2}{n-1},\ S \ni 1,\ s = n-1\}$$

and are in one-to-one correspondence with elements of $\mathbf{X}$.

Theorem 3. *Let $\nu \in G^N$. Then* *(i)* $\nu \in ext_{NI}(\mathbf{SP}^n)$ *iff there is such* $x \in \mathbf{X}$ *that* $\nu = \nu^x = \Psi^{-1}(\bar{\nu}^x)$*, where* $\bar{\nu}^x(S) = f_s^x$ *for all* $S \in \Omega$ *and*

$$f_s^x = \begin{cases} o, & s = 2,\ x_2 = 0, \\ \frac{n-2}{n-1}, & s = n-1, \\ f_{s-1}^x, & s \geq 3,\ x_s = 0, \\ \frac{s-1}{n-1}, & x_s = 1, \end{cases}$$

(ii) $\nu \in ext_I(\mathbf{SP}^n)$ *iff there is such* $k \in \{2, \ldots, n-1\}$ *that* $\nu = \nu^k$*, where* ν^k *was determined in Proposition 1.*

Proof. (i) Suppose $\nu \in \mathbf{SP}^n$ and $\bar{\nu} = \Psi(\nu)$. Then $\bar{\nu}(S) = f(|S|) = f_s$, $S \in \Omega$. So, system (7)-(10) takes the form

$$\left.\begin{array}{ll} f_2 \geq 0,\ \ \frac{n-2}{n-1} \leq f_{n-1} \leq 1, & \\ f_{s-1} \leq f_s, & s \in J \setminus 2, \\ f_s - (n-s)f_{n-1} \leq s+1-n, & s \in J. \end{array}\right\}$$

Let $\mathbf{F}^n$ be the polytope specified by this system. Take $\hat{f} \in \mathbf{R}^{n-2}$ with

$$\hat{f}_s = \begin{cases} \frac{n-1}{n}, & s = n-1, \\ \frac{s-1}{n}, & s \in J. \end{cases}$$

Since $\mathbf{F}^n \subset \mathbf{R}^{n-2}$ and $\hat{f}$ is the interior point of $\mathbf{F}^n$ then $dim(\mathbf{F}^n) = |J| = n-2$. For each $x \in \mathbf{X}$, $f^x \in \mathbf{F}^n$ and satisfies following $(n-2)$ equations

$$\left.\begin{array}{ll} f_2 = 0\ \ if\ \ x_2 = 0, & f_{n-1} = \frac{n-2}{n-1}, \\ f_s = \frac{s-1}{n-1}\ \ if\ x_s = 1, & f_{s-1} = f_s\ \ if\ s \geq 3\ and\ x_s = 0. \end{array}\right\} \tag{14}$$

Obviously, system (14) has the unique solution. This implies

$$f^x \in ext_{NI}(\mathbf{F}^n) \implies \bar{\nu}^x \in ext_{NI}(\bar{\mathbf{P}^n}) \implies \nu^x \in ext_{NI}(\mathbf{P}^n).$$

To prove inverse, take $f' \in ext_{NI}(\mathbf{F}^n)$. We shall show in the beginning that $f'_{n-1} = \frac{n-2}{n-1}$. Suppose $\frac{n-2}{n-1} < f'_{n-1} < 1$. For each $s \in J \setminus 2$ denote

$$k_s = max\{k \in J:\ f'_k < f'_s\}$$

if there is such $k \in J$ that $f'_k < f'_s$. From monotonicity conditions follows that $k_s < s$. Obviously, exists $\delta > 0$ satisfying the inequalities

$$\delta \leq f'_{n-1} - \frac{n-2}{n-1}, \quad \delta \leq 1 - f'_{n-1},$$

$$\delta \leq \frac{f'_s}{n-s}\ \ for\ f'_s > 0,\ \ \delta \leq \frac{f'_s - f'_{k_s}}{s - k_s}\ \ for\ f'_s > f'_{k_s},\ \ s \in J.$$

Consider vectors f_s^-, f_s^+ determined by

$$f_2^- = \begin{cases} 0, & f'_2 = 0, \\ f'_2 - (n-2)\delta, & f'_2 > 0, \end{cases} \qquad f_2^+ = \begin{cases} 0, & f'_2 = 0, \\ f'_2 + (n-2)\delta, & f'_2 > 0, \end{cases}$$

$$f_s^- = \begin{cases} f'_{n-1}, & s = n-1, \\ f'_s - (n-s)\delta, & f'_s > f'_{s-1}, \\ f_{s-1}^-, & f'_s = f'_{s-1}, \end{cases} \qquad f_s^+ = \begin{cases} f'_{n-1}, & s = n-1, \\ f'_s + (n-s)\delta, & f'_s > f'_{s-1}, \\ f_{s-1}^-, & f'_s = f'_{s-1}, \end{cases}$$

$s \in J$. From the definition of δ and the fact that $\frac{n-2}{n-1} > \frac{n-s-1}{n-s}$ for all $s \in J$, follows $\delta \leq f'_{n-1} - \frac{n-s-1}{n-s}$, $s \in J$. Moreover, $f'_s - (n-s)f'_{n-1} < s+1-n$ if $f'_s = f'_{s-1} > 0$, $s \in J \setminus 2$, because otherwise we have

$$\left.\begin{array}{l} f'_s - (n-s)f'_{n-1} = s+1-n, \\ f'_{s-1} - (n-s+1)f'_{n-1} \leq s-n, \\ f'_s = f'_{s-1}, \end{array}\right\} \implies f'_{n-1} \geq 1,$$

which contradicts the assumption $f'_{n-1} < 1$. Tables 3,4 show that $f^-,\ f^+ \in \mathbf{F}^n$. The equality $f' = \frac{f^- + f^+}{2}$ implies $f' \notin ext(\mathbf{F}^n)$. Thus, all non-integer extreme points of $\mathbf{F}^n$ belongs to its facet determined by constraints

$$f_{n-1} = \frac{n-2}{n-1},\ f_2 \geq 0,\quad f_{s-1} \leq f_s\ \ if\ s \in J \setminus \{2\},\ f_s \leq \frac{s-1}{n-1}\ \ if\ s \in J. \tag{15}$$

Since the constraints matrix of system (15) is totally unimodular and $f' \in ext(\mathbf{F}^n)$, the values f'_s, $s \in J \setminus (n-1)$, can be equal to 0 or $\frac{1}{n-1}$ for $s = 2$ and $\frac{s-1}{n-1}$ or f_{s-1} for $s \in J \setminus \{2\}$, i.e. f' must be coincides with f^x_s for some $x \in \mathbf{X}$.

Item (ii) is proved analogously. □

Table3: Representation f^- through f'.

Cases	f^-_s	f^-_{s-1}	$f^-_s - (n-s)f^-_{n-1}$
$s=2,\ f'_2 = 0,$	0	–	$-(n-2)(f'_{n-1} - \delta),$
$s=2,\ f'_2 > 0,$	$f'_2 - (n-2)\delta,$	–	$f'_2 - (n-2)f'_{n-1},$
$s>3, f'_s = f'_{s-1} = 0,$	0	f^-_s	$-(n-s)(f'_{n-1} - \delta),$
$s>3, f'_s = f'_{s-1} > 0,$	$f'_{k_s} - (n-k_s)\delta,$	f^-_s	$f'_{k_s} - (n-s)f'_{n-1} + (s-k_s)\delta,$
$s>3, f'_s > f'_{s-1} = 0,$	$f'_s - (n-s)\delta,$	0	$f'_s - (n-s)f'_{n-1},$
$s>3, f'_s > f'_{s-1} > 0,$	$f'_s - (n-s)\delta,$	$f'_{k_s} - (n-k_s)\delta,$	$f'_s - (n-s)f'_{n-1}$

Table4: Representation f^+ through f'.

Cases	f_s^+	f_{s-1}^+	$f_s^+ - (n-s)f_{n-1}^+$
$s=2,\ f_2'=0,$	0	$-$	$-(n-2)(f_{n-1}'+\delta),$
$s=2,\ f_2'>0,$	$f_2'+(n-2)\delta,$	$-$	$f_2'-(n-2)f_{n-1}',$
$s>3, f_s'=f_{s-1}'=0,$	0	f_s^-	$-(n-s)(f_{n-1}'+\delta),$
$s>3, f_s'=f_{s-1}'>0,$	$f_{k_s}'+(n-k_s)\delta,$	f_s^-	$f_{k_s}'-(n-s)f_{n-1}'-(s-k_s)\delta,$
$s>3, f_s'>f_{s-1}'=0,$	$f_s'+(n-s)\delta,$	0	$f_s'-(n-s)f_{n-1}',$
$s>3, f_s'>f_{s-1}'>0,$	$f_s'+(n-s)\delta,$	$f_{k_s}'+(n-k_s)\delta,$	$f_s'-(n-s)f_{n-1}'$

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Stackelberg Tariff Games between Provider, Primary and Secondary Users

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Abstract Due to the development of wireless communication technologies and the increase of integrated wireless networks, the problem of spectrum bandwidth management has become a hot research field in recent years. In this paper, we consider this problem as hierarchical game with three types of players: spectrum holder (provider), primary and secondary user. Provider assigns the price for bandwidth resource to maximize its own profit, whereas the primary user tries to find balance between using the bandwidth resource for its own needs and renting spectrum band to secondary users for profit gain. Secondary users can access to the network and transmit their signals by using the spectrum band of the primary user paying for that in proportion to the power of the transmitted signal. For this game the optimal strategies of the players are found. Numerical modelling demonstrates how the equilibrium strategies and corresponding payoffs depend of network parameters.

Keywords: spectrum market, bandwidth resource, Stackelberg game, power control.

1. Introduction

Bandwidth resource management is an important issue in wireless networks. Spectrum, which is a scarce resource, is comparatively efficiently used in wireless technologies such as WiMAX and UMTS. The quality of service (QoS) reached by a user, which is communicating a signal, heavily depends on the value of bandwidth provided by a spectrum holder.

From economic point of view, holders and users (primary, secondary, etc) form a spectrum market, within which spectrum bands are considered as a resource flowing from owners to consumers regulated by market mechanisms. Users are able to rent the spectrum band purchased from the holders. Thus, for holders it is important to find the optimal price to sell the band spectrum, whereas users try to find balance between using the bandwidth resource for its own needs and renting unused bandwidth capacity for profit gain.

The problem of such hierarchical spectrum bandwidth sharing has become a hot research field in recent years due to the development of wireless communication technologies and the increase of integrated wireless networks. In the study (Niyato and Hossain, 2007), the network model, when a WiMAX base station serves both WiMAX subscriber stations and WiFi access points/routers in its coverage area, is considered. The WiMAX and WiFi service providers try to maximize its own profits by defining the optimal price on spectrum bands. This problem is formulated as a Stackelberg leader-follower game in which the WiMAX base station and the WiFi routers are the leader and the followers, respectively. The network which is comprised by two-level architecture (WiMAX and UMTS networks serve WLAN as backbone) is considered in (Ming et al., 2009), and Vickrey-Clarke-Groves auction mechanism is used to allocate the bandwidth resource efficiently among various entities in this network. An access pricing schemes for multi-hop wireless communications are investigated in (Lam et al., 2006). The pricing is used as an incentive for the relay node to forward traffic to the gateway. A game-theoretic approach is used to analyze interactions of the access point, wireless relaying nodes, and clients from one-hop to multihop networks and when the network has an unlimited or limited channel capacity. Three level pricing scheme authority-provider-user where tariff of access to internet is proportional to throughput is studied in (Garnaev et al., 2010).

In this paper we introduce the following hierarchical game with three different types of players: spectrum holder (provider), a primary and several secondary users. The provider assigns the price for bandwidth resource to maximize its own profit. The primary user buys a license for using frequency bandwidth from the provider to transmit signals. The primary user can also earn some extra money giving unused frequencies capacity for rent to secondary users charging them by assigned tariff for interference. Each secondary user can either buy the frequency bandwidth or it can choose the service of the other provider based on comparing of the suggested QoS and prices which is set for the network access. For this game the optimal strategies of the players are found. Numerical modelling demonstrates how the equilibrium strategies and corresponding payoffs depend of network parameters.

The rest of the article is organized as follows. Section 2. presents system model for hierarchical spectrum sharing considered in this paper. The optimal strategies of the players are found in Section 3.. After that dependence of the equilibrium strategies and corresponding payoffs on network parameters is presented in Section 4.. Section 5. concludes the paper.

2. System model and assumptions

In this section, we present the market model and corresponding assumptions. Let us consider a case in which there is only one the frequency spectrum provider, one primary user which purchases a spectrum band from the provider and several secondary users which can pay for the possibility to use the spectrum of the primary user.

As a rule, in such the spectrum markets the provider is large commercical or governmental organization which possess the rights on the spectrum license and try to share this spectrum optimally. On the other hand, primary user is service distributor which provides the access to the network for end subscribers - secondary users. The provider is trying to maximize its own profit by assigning the optimal price for the bandwidth resource, whereas users are maximizing the quality of services which they provide by utilizing this resource taking into account the resource costs.

We formulate this problem as the hierarchical game with three types of players: a provider, a primary and a secondary user. The provider sets the price for bandwidth resource to maximize its own profit. Thus, the provider's payoff π^{prov} is the profit obtained by selling the bandwidth resource to the primary user:

$$\pi^{prov}(C_W) = C_W W, \tag{1}$$

where $W \geq 0$ is the spectrum band bought by the primary user and $C_W > 0$ is its price assigned by the provider. Thus, the strategy of the provider is to choose C_W, which is unbounded non-negative value.

The primary user purchases a spectrum band directly from the provider, afterwards it allows several secondary users to use the spectrum band so that to maximize its own payoff. The payoff of the primary user π^P is its throughput plus profit it obtains by giving access to the network for secondary users minus how much it costs to buy the bandwidth resource from the holder. Let us assume that the primary user charges secondary users proportional to their interfering power:

$$\pi^P(W, C_P) = \alpha W \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W + \sum_{i=1}^n h_i^S P_i^S}\right) - C_W W + C_P h_i^S P_i^S, \tag{2}$$

where $\alpha > 0$ is coefficient which shows the economic efficiency of throughput for the primary user, $G^P > 0$ is is the spreading gain of the CDMA system for the primary user, $h^P > 0$ and $h_i^S > 0$ are the fading channel gains for the primary and the i-th secondary user respectively, $\sigma^2 \geq 0$ is the background noise, $P_P \geq 0$ is the transmitting power employed by the primary user, $C_P \geq 0$ is the tariff for access to the network for the secondary user, $P_i^S \geq 0$ is the power employed by the i-th secondary user. The primary user is trying to maximize (2) by finding optimal W and C_P.

Similarly, a secondary user payoff is the throughput value provided by it minus bandwidth resource costs, i.e.

$$\pi_i^S(P^S) = \beta W \log\left(1 + G^S \frac{h_i^S P_i^S}{\sigma^2 W + h^P P^P + \sum_{j \neq i} h_j^S P_j^S}\right) - C_P h_i^S P_i^S, \tag{3}$$

where $P^S = (P_1^S, \dots, P_n^S)$ is vector of secondary users transmitting powers, $\beta > 0$ is coefficient which shows the economic efficiency of throughput for each secondary user and $G^S > 1$ is is the spreading gain of the CDMA system for secondary users. The i-th secondary user is looking for non-negative $P_i^S \geq 0$ which maximizes its own payoff. Let us assume that maximal power which can be used by secondary users to transmit the signals is very high and in this study we will not take into account the upper limit for secondary users transmitting power.

Let us consider the problem defined as a sequence of two Stackelberg games (Romp, 1997). In the first stage, the provider maximizes its own profit by assigning the price for bandwidth resource, and the primary user buys a license for using frequency bandwidth from the provider to transmit own data without taking into account secondary users:

$$\max_{C_W} \pi^{prov}(C_W) = \max_{C_W} C_W W,$$
subject to $C_W > 0$.
$$\max_{W} \pi^{P}(W) = \max_{W} \alpha W \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W}\right) - C_W W,$$
subject to $W \geq 0$.

In the second stage, the primary user allows secondary users to use the spectrum band purchased in order to increase its own prifit. We assume that the spectrum band is fixed and the primary user charges secondary users proportional to their interfering power. Each secondary user transmits its own signal in such power mode that maximizes its payoff function:

$$\max_{C_P} \pi^P(W, C_P) =$$
$$= \max_{W, C_P} \alpha W \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W + \sum_{i=1}^n h_i^S P_i^S}\right) - C_W W + C_P \sum_{i=1}^n h_i^S P_i^S,$$
subject to $C_P > 0$.
$$\max_{P^S} \pi_i^S(P^S) = \max_{P^S} \beta W \log\left(1 + G^S \frac{h_i^S P_i^S}{\sigma^2 W + h^P P^P + \sum_{j=1, j\neq i}^n h_j^S P_j^S}\right) - C_P h_i^S P_i^S,$$
subject to $P_i^S \geq 0, \forall i \in \{1, \dots, n\}$.

3. Solution of the game

3.1. Game between the provider and the primary user

First, let us consider the case when there are only the provider and the primary user: the provider maximizes its own profit by assigning the optimal price for frequency band:

$$\max_{C_W} \pi^{prov}(C_W) = \max_{C_W} C_W W, \quad \text{subject to } C_W > 0. \tag{4}$$

and the primary user maximizes its own payoff by purchasing the optimal amount of this band:

$$\max_{W} \pi^{P}(W) = \max_{W} \alpha W \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W}\right) - C_W W, \quad \text{subject to } W \geq 0. \tag{5}$$

The following theorem allows to obtain the Nash equilibrium of the frequency spectrum game (4,5).

Theorem 1. *Under the assumptions made in the previous section, the spectrum band game between the provider and the primary user (4,5) admits a unique Nash equilibrium (NE). The corresponding to that NE optimal spectrum band price assigned by the provider C_W^* can be found as unique root of following equation:*

$$(L(-e^{-\frac{C_W}{\alpha}-1}) + 1)^2 = \frac{C_W}{\alpha}. \tag{6}$$

and optimal spectrum band bought by the primary user W^* can be calculated as follows:

$$W^* = -\frac{G^P h^P P^P L\left(-e^{-\frac{C_W^*}{\alpha}-1}\right)}{\sigma^2\left(1+L\left(-e^{-\frac{C_W^*}{\alpha}-1}\right)\right)}. \tag{7}$$

Proof. In order to solve for the game (4,5) we use a backward induction technique. We start with the optimization problem of the primary user and derive the best response for this user as a function of the price C_W set by the service provider.

The objective function in (5) is continuously differentiable concave and inequality constraint is continuously differentiable convex function. Therefore, Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality, i.e. the frequency band W is optimal if and only if the following conditions are satisfied:

$$C_W + \alpha\frac{G^P h^P P^P}{\sigma^2 W + G^P h^P P^P} - \alpha\log\left(1+\frac{G^P h^P P^P}{\sigma^2 W}\right)\begin{cases}\geq 0, W = 0,\\ = 0, W \geq 0.\end{cases} \tag{8}$$

For any finite $C_W > 0$ the optimal frequency band purchased by the primary user from the provider can be found as the following reaction function:

$$W(C_W) = -\frac{G^P h^P P^P L\left(-e^{-\frac{C_W}{\alpha}-1}\right)}{\sigma^2\left(1+L\left(-e^{-\frac{C_W}{\alpha}-1}\right)\right)}, \text{ if } C_W < +\infty, \tag{9}$$

where $L(x)$ is Lambert function which is implicit function which is defined by the following equation:

$$x = L(x)e^{L(x)}. \tag{10}$$

In the case of finite C_W, the argument of Lambert function is on the following interval $(-e^{-1}, 0)$, i.e. it is at least greater than $[-e^{-1}$. In order to satisfy the constraint $W \geq 0$, the Lambert function values have to be located on the interval $[-1, 0]$, i.e. $L\left(-e^{-\frac{C_W}{\alpha}-1}\right)$ is at least greater or equal to zero. Thus, we can restrict to single-valued Lambert function with real values.

Thus, the provider profit as a function of spectrum band price C_W can be found as follows:

$$\pi_{prov}(C_W) = C_W W(C_W) = -\frac{G^P h^P P^P C_W L\left(-e^{-\frac{C_W}{\alpha}-1}\right)}{N\left(1+L\left(-e^{-\frac{C_W}{\alpha}-1}\right)\right)}. \tag{11}$$

This function is continuous function of C_W. When $0 < C_W < \infty$, this function is increasing when $(L(-e^{-\frac{C_W}{\alpha}-1})+1)^2 \geq \frac{C_W}{\alpha}$, and it is decreasing when $(L(-e^{-\frac{C_W}{\alpha}-1})+1)^2 \leq \frac{C_W}{\alpha}$, i.e. this function reaches its maximal value when the following condition is satisfied:

$$(L(-e^{-\frac{C_W}{\alpha}-1})+1)^2 = \frac{C_W}{\alpha}. \tag{12}$$

Since $L(x)$ is strictly increasing for $x > -e^{-1}$, the left part of this equation is continuous increasing concave function on the interval $0 < C_W < \infty$. In addition, it is equal to zero at the point $C_W = 0$ and converges to 1 when $C_W \to \infty$. Thus, the equation $eq1$ can not have more than one root. We can easily check that the value of the function $(L(-e^{-\frac{C_W}{\alpha}-1}) + 1)^2|_{C_W=0.4\alpha} > \frac{C_W}{\alpha}|_{C_W=0.4\alpha}$. On the other side, $(L(-e^{-\frac{C_W}{\alpha}-1}) + 1)^2|_{C_W=\alpha} < \frac{C_W}{\alpha}|_{C_W=\alpha}$. Thus, there is a unique root C_W^* of (12) for $C_W > 0$ and it lies in the interval $0.4\alpha < C_W^* < \alpha$. This root C_W^* corresponds the maximal value of the provider profit. Let us notice that optimal price which assigned by the provider does not depend on average received power to noise power spectral density ratio. We can easily find C_W^* numerically.

The optimal spectrum band bought by the primary user W^* can be calculated according to the reaction function (9) $W^* = W(C_W^*)$. Obtained solution (W^*, C_W^*) is NE point of the game (4,5) found by backward induction. This point is unique since the reaction function (9) is single-valued and the root of the equation (12) is unique.

3.2. Game between primary user and secondary users

Let us assume that after finding optimal price C_W of frequency band, the provider does not care about how the primary user uses purchased frequency band, i.e. C_W found from previous section is fixed. The primary user allows secondary users to use purchased frequency band and assigns the price in proportion to the power of the transmitted signals of secondary users. Thus, the following two-stage game is considered. The primary user buys a license for using frequency bandwidth from the provider to transmit signals and rent frequency band to the secondary user so that to maximize its own payoff:

$$\max_{W,C_P} \pi^P(W, C_P) =$$
$$= \max_{W,C_P} \alpha W \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W + \sum_{i=1}^n h_i^S P_i^S}\right) - C_W W + C_P \sum_{i=1}^n h_i^S P_i^S, \tag{13}$$
$$\text{subject to } \begin{cases} W \geq 0, \\ C_P > 0. \end{cases}$$

Each secondary user tries to maximize its own payoff by using the primary user frequency band:

$$\max_{P^S} \pi_i^S(P^S) = \max_{P^S} \beta W \log\left(1 + G^S \frac{h_i^S P_i^S}{\sigma^2 W + h^P P^P + \sum_{j=1, j\neq i}^n h_j^S P_j^S}\right) - C_P h_i^S P_i^S,$$
$$\text{subject to } P_i^S \geq 0, \forall i \in \{1, \ldots, n\}. \tag{14}$$

First, let us consider the optimization problem of finding optimal transmitting powers for secondary users (14). The objective functions in (14) are continuously differentiable concave and inequality constraints are continuously differentiable convex functions. Therefore, Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality, i.e. the transmitting powers P_i^S are optimal if and only if the following conditions are satisfied for any $i \in \{1, \ldots, n\}$:

$$\frac{\beta W G^S h_i^S}{\sigma^2 W + h^P P^P + \sum_{j=1, j\neq i}^n h_j^S P_j^S + G^S h_i^S P_i^S} - C_P h_i^S \begin{cases} \leq 0, & \text{if } P_i^S = 0, \\ = 0, & \text{if } P_i^S \geq 0, \end{cases} \tag{15}$$

The optimal strategy of the i-th secondary user can be found from (15) as follows:

$$P_i^{S^*} = \begin{cases} \frac{1}{h_i^S}\left(\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S} - \frac{\sum_{j=1,j\neq i}^n h_j^S P_j^S}{G^S}\right), & \text{if } C_P \in I_{i1}, \\ 0, & \text{if } C_P \in I_{i0}, \end{cases} \tag{16}$$

where

$$\begin{aligned} I_{i0} &= \left[\frac{\beta W G^S}{\sigma^2 W + h^P P^P + \sum_{j=1,j\neq i}^n h_j^S P_j^S}, +\infty\right), \\ I_{i1} &= \left(0, \frac{\beta W G^S}{\sigma^2 W + h^P P^P + \sum_{j=1,j\neq i}^n h_j^S P_j^S}\right). \end{aligned} \tag{17}$$

In order for the i-th secondary user to transmit a signal, i.e. $P_i^S > 0$ the following conditions from (16) have to hold:

$$\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S} > 0 \tag{18}$$

and

$$\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S} - \frac{\sum_{j=1,j\neq i}^n h_j^S P_j^S}{G^S} > 0. \tag{19}$$

An intuitive interpretation for these conditions is the following: if the price C_P is set too high for a secondary user, this secondary user prefers not to transmit at all, depending on its channel gain, utility parameter, the spreading gain, etc. In equation (16), the i-th secondary user transmitting power depends not only on the user-specific parameters, like h_i^S, but also on the network parameter G^S, and total power level received by the primary user base station $\sum_{i=1}^n h_i^S P_i^S$.

For any equilibrium solution, the set of fixed point equations can be written in matrix form. The rows and columns corresponding to users with zero equilibrium power are deleted, and the equation below involves only the users with positive powers. It is obvious that if condition (18) does not hold none of the secondary users does not transmit any signals. Assuming that (18) is satisfied and all n users have positive power levels at equilibrium, we have the following equation for finding oprimal secondary users powers $P^{S*} = (P_1^{S*}, P_2^{S*}, \ldots, P_n^{S*})$:

$$\begin{pmatrix} 1 & \frac{h_2^S}{h_1^S G^S} & \frac{h_3^S}{h_1^S G^S} & \cdots & \frac{h_n^S}{h_1^S G^S} \\ \frac{h_1^S}{h_2^S G^S} & 1 & \frac{h_3^S}{h_2^S G^S} & \cdots & \frac{h_n^S}{h_2^S G^S} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{h_1^S}{h_n^S G^S} & \frac{h_2^S}{h_n^S G^S} & \frac{h_3^S}{h_n^S G^S} & \cdots & 1 \end{pmatrix} \begin{pmatrix} P_1^{S*} \\ P_2^{S*} \\ \vdots \\ P_n^{S*} \end{pmatrix} = \begin{pmatrix} \frac{1}{h_1^S}\left(\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S}\right) \\ \frac{1}{h_2^S}\left(\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S}\right) \\ \vdots \\ \frac{1}{h_n^S}\left(\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S}\right) \end{pmatrix}. \tag{20}$$

We denote the vector in the right side as $c = (c_1, c_2, \ldots, c_n)^T = \left(\frac{a_1}{h_1^S}, \frac{a_2}{h_2^S}, \ldots, \frac{a_n}{h_n^S}\right)^T$. Let us state the following proposition by adopting the results in (Alpcan et al., 2001):

Theorem 2. *In the defined power game with n users, let the indexing be done such that $i > j$ if $a_i < a_j$, with the ordering picked arbitrarily if $a_i = a_j$. Let $m^* \leq n$ be* *the largest integer m for which the following condition is satisfied:*

$$a_m > \frac{1}{G^S + m - 1} \sum_{i=1}^{m} a_i. \tag{21}$$

Then the power game admits a unique Nash equilibrium, which has the property that users $m^* + 1, \dots, n$ have zero power levels, i.e.

$$P_i^S = 0, \text{ if } i \in \{m^* + 1, \dots, n\}. \tag{22}$$

The equilibrium power levels of the first m^* secondary users can be calculated uniquely from (20) as follows:

$$P_i^{S*} = \frac{G^S}{h_i^S(G^S - 1)} \left(a_i - \frac{\sum_{j=1}^{m^*} a_j}{G^S + m^* - 1} \right). \tag{23}$$

One can easily notice that in the game between the primary user and n secondary users (13,14) all variables a_i are equal to each other, i.e. $a_1 = a_2 = \ldots = a_n = \frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S}$. Therefore inequality (21) can be rewritten as follows:

$$\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S} > \frac{1}{G^S + m - 1} \sum_{i=1}^{m} \left(\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S} \right). \tag{24}$$

Since we assumed that the spreading gain of the CDMA system for secondary users $G^S > 1$, the inequality (24) is satisfied for any $1 \leq m \leq n$, and the largest such m is equal to n. Thus, we can formulate the following corollary which defines optimal strategies of secondary users:

Corollary 3.1. *Under the assumption that $G^S > 1$, in the game between the primary user and n secondary users (13,14) secondary users have following optimal strategies depending on the price assigned by the primary user:*

$$\begin{aligned} &\text{If } 0 \leq C_P < \frac{\beta W G^S}{\sigma^2 W + h^P P^P}, \text{ then } P_i^{Sopt}(C_P) = \frac{G^S \left(\frac{\beta W}{C_P} - \frac{\sigma^2 W + h^P P^P}{G^S} \right)}{h_i^S (G^S + n - 1)}, \\ &\text{If } C_P \geq \frac{\beta W G^S}{\sigma^2 W + h^P P^P}, \text{ then } P_i^{Sopt}(C_P) = 0, \end{aligned} \tag{25}$$

for any $i \in \{1, \dots, n\}$.

The i-th secondary user payoff can be calculated as follows:

$$\begin{aligned} \pi_i^{S*}(C_P) &= \beta W \log \left(\frac{\beta W (G^S + n - 1)}{C_P(\sigma^2 W + h^P P^P) + (n-1)\beta W} \right) - \\ &- \frac{G^S \beta W - C_P(\sigma^2 W + h_P P_P)}{G^S + n - 1}, \text{if } 0 \leq C_P < \frac{\beta W G^S}{\sigma^2 W + h^P P^P}, \\ \pi_i^{S*}(C_P) &= 0, \text{if } C_P \geq \frac{\beta W G^S}{\sigma^2 W + h^P P^P}. \end{aligned} \tag{26}$$

If secondary users apply their optimal strategies the payoff of the primary user can be rewritten in the following form:

$$\begin{aligned}
&\pi^{P*}(C_P,W)=\alpha W\log\left(1+\frac{(G^S+n-1)G^Ph^PP^P}{\sigma^2W(G^S-1)-nh^PP^P+\frac{1}{C_P}nG^S\beta W}\right)-\\
&-C_P\frac{n(\sigma^2W+h^PP^P)}{G^S+n-1}+W\left(\frac{nG^S\beta}{G^S+n-1}-C_W\right),\ \text{if } 0\le C_P<\frac{\beta WG^S}{\sigma^2W+h^PP^P},\\
&\pi^{P*}(C_P,W)=\alpha W\log\left(1+G^P\frac{h^PP^P}{\sigma^2W}\right)-C_WW,\ \text{if } C_P\ge\frac{\beta WG^S}{\sigma^2W+h^PP^P}.
\end{aligned}\tag{27}$$

Let us introduce the following notations:

$$\begin{aligned}
&a:=(G^S+n-1)G^Ph^PP^P,\quad b:=\sigma^2W(G^S-1)-nh^PP^P,\quad c:=nG^S\beta W,\\
&d:=\frac{n(\sigma^2W+h^PP^P)}{\alpha W(G^S+n-1)},\quad \bar{C}_P=\frac{\beta WG^S}{\sigma^2W+h^PP^P},\\
&\bar{P}_i^S(C_P)=\frac{G^S\left(\frac{\beta W}{C_P}-\frac{\sigma^2W+h^PP^P}{G^S}\right)}{h_i^S(G^S+n-1)},\forall i\in\{1,\dots,n\},\\
&\bar{\pi}^P(C_P)=\alpha W\log\left(1+\frac{(G^S+n-1)G^Ph^PP^P}{\sigma^2W(G^S-1)-nh^PP^P+\frac{1}{C_P}nG^S\beta W}\right)-\\
&-C_P\frac{n(\sigma^2W+h^PP^P)}{G^S+n-1}+W\left(\frac{nG^S\beta}{G^S+n-1}-C_W\right),\ \text{if } 0\le C_P<\frac{\beta WG^S}{\sigma^2W+h^PP^P}.
\end{aligned}\tag{28}$$

The following theorem defines the conditions when the game between the primary user and secondary users (13,14) has inner NE point, i.e. $C_P>0$ and $P_i^S>0,\forall i\in\{1,\dots,n\}$.

Theorem 3. *If parameters defined in (28) satisfy one of the following sets of conditions:*

$$\begin{cases}
b<0,\\
4b^2+4ab+acd>0,\\
0<\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}<\bar{C}_P,\\
\bar{\pi}^P\left(\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}\right)<\bar{\pi}^P\left(\bar{C}_P\right),
\end{cases}\tag{29}$$

$$\begin{cases}
b=0,\\
0<\frac{1}{d}-\frac{c}{a}<\bar{C}_P,
\end{cases}\tag{30}$$

or

$$\begin{cases}
b>0,\\
0<\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}<\bar{C}_P,
\end{cases}\tag{31}$$

then the game between the primary user and n secondary users (13,14) admits a unique inner NE point (C_P^*, P^{S*}) *such that*

$$\begin{aligned} &\text{If conditions (30) hold, then } \begin{cases} C_P^* = \frac{1}{d} - \frac{c}{a}, \\ P_i^{S*} = \bar{P}_i^S(C_P^*), \forall i \in \{1, \dots, n\} \end{cases} \\ &\text{If conditions (29) or (31) hold, then } \begin{cases} C_P^* = \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}, \\ P_i^{S*} = \bar{P}_i^S(C_P^*), \forall i \in \{1, \dots, n\} \end{cases} \end{aligned} \tag{32}$$

Proof. The proof of the theorem is deferred to the Appendix.

Corollary 3.2. *If the following set of conditions*

$$\begin{cases} b < 0, \\ 4b^2 + 4ab + acd > 0, \\ 0 < \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)} < \bar{C}_P, \\ \bar{\pi}^P\left(\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}\right) = \bar{\pi}^P\left(\bar{C}_P\right), \end{cases} \tag{33}$$

is fulfilled, then

$$C_P^* = 0 \ \text{or} \ \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)},$$

and each secondary user transmission power is $P_i^{S*} = P_i^{Sopt}(C_P^*)$.

Corollary 3.3. *If none of the conditions (29), (30), (31) or (33) is fulfilled, then the optimal price* C_P^* *assigned by the primary user can be calculated as follows*

$$\begin{cases} C_P^* = 0, \ \text{if} \ \frac{nG^S\beta}{G^S+n-1} > \alpha \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W}\right), \\ C_P^* = C_P^{**}, \ \text{if} \ \frac{nG^S\beta}{G^S+n-1} < \alpha \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W}\right), \\ C_P^* = 0 \ \text{or} \ C_P^{**}, \ \text{if} \ \frac{nG^S\beta}{G^S+n-1} = \alpha \log\left(1 + G^P \frac{h^P P^P}{\sigma^2 W}\right), \end{cases}$$

where C_P^{**} *is any* C_P *from the interval* $[\bar{C}_P, +\infty)$. *In this case, each secondary user transmitting power is* $P_i^{S*} = P_i^{Sopt}(C_P^*)$.

Let us conclude that the Theorem 3 and its corollaries allow to calculate the optimal price assigned by the primary user. Thus, after buying optimal spectrum band from the provider, the primary user allows secondary users to transmit their signals in the same spectrum band and assigns the price C_P^*. The solution $C_P = 0$ means that provider aims to assign the minimum possible price whereas secondary users transmission power values are maximum possible, i.e. $P_i^{S*} \to +\infty$. When $C_P^* \geq \bar{C}_P$, the primary user in fact forbids secondary users to transmit signals in the same spectrum band.

4. Numerical examples

The provider payoff $\pi^{prov}(C_W)$ for $\alpha = 1$ and different values of average received power to noise power spectral density ratio are shown in Fig. 1. The optimal value of C_W is equal to 0.468. One can notice that the payoff of the provider does not depend on network parameters but depends only on the primary user willingness to pay factor. The function π^P for different values of average received power to noise power spectral density ratio when C_W is equal to optimal is shown in Fig. 2. Thus, the primary user grows when its SINR increases.

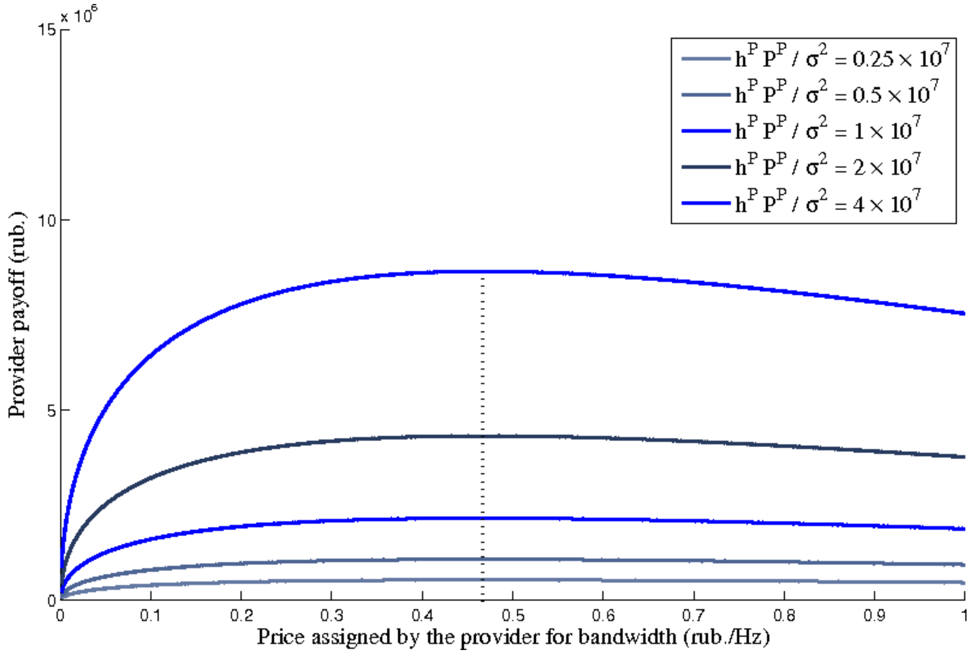

Figure1: Provider profit for different values of average received power to noise power spectral density ratio.

Figure 3 depicts how the primary user payoff depends on the price assigned by him as a fee for secondary users to transmit signals in the same spectrum band. For all examples considered, the optimal strategy of the primary user is to assign very high price so that secondary users prefer not to transmit their signals in the primary user frequency band.

5. Conclusion

In this paper, we have introduced the hierarchical game between provider, a primary and several secondary users. The provider assigns the price for bandwidth resource to maximize its own profit, whereas the primary user buys a license for using frequency bandwidth from a provider to transmit signals. The primary user can also earn some extra money giving unused frequencies capacity for rent to secondary users charging them by assigned tariff for interference and each secondary user can either stay inactive or transmit a signal using the frequency band rented from the primary user. We have considered this hierarchical game as a sequence of two Stackelberg games: one between the provider and the primary user, and one between the primary user and secondary users. For each such game optimal strate-

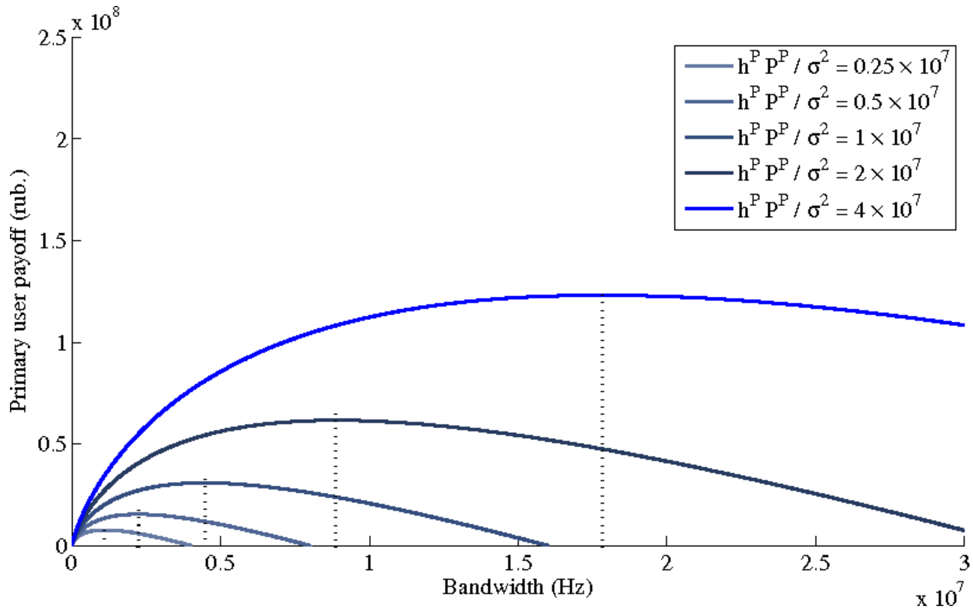

Figure2: Primary user payoff for different values of average received power to noise power spectral density ratio when C_W is optimal.

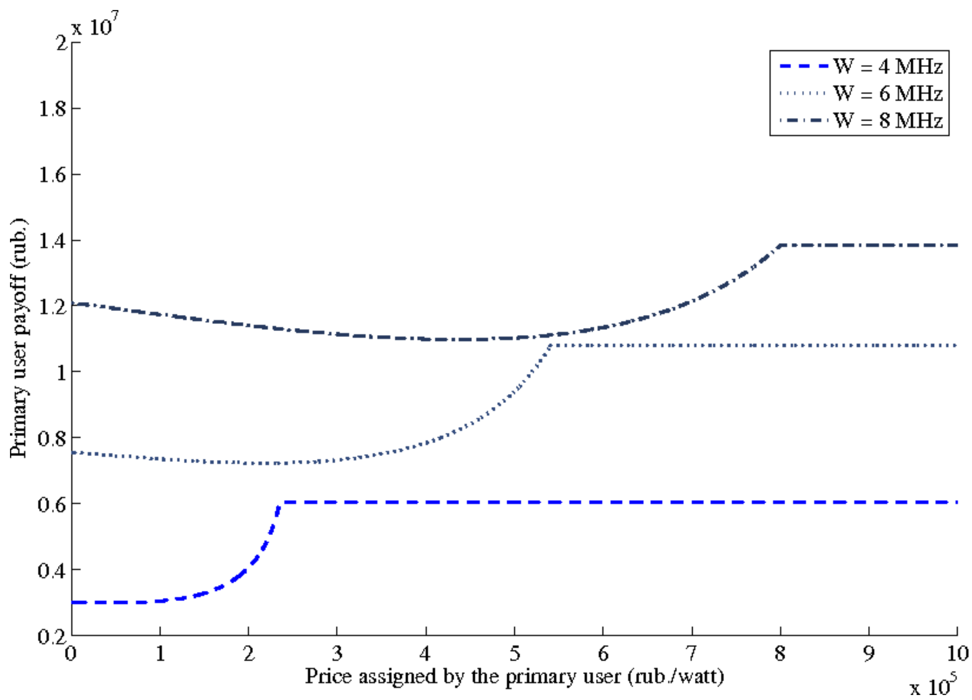

Figure3: Primary user payoff for different values of the spectrum band purchased.

gies of the players have been found. Numerical modelling has demonstrated how the equilibrium strategies and corresponding payoffs depend of network parameters.

In the future, we are planning to consider also the case when the provider takes into account activity of secondary users when it sells the spectrum band to the primary user. There is possibility when it is beneficial for the primary user to inform the provider that secondary users transmit their signals in the same spectrum band.

Appendix

In this section we give a proof of Theorem 3:

Proof. Based on the assumptions listed in Section 2. we can easily see that

$$\begin{aligned} & a>0, \quad c>0, \quad d>0, \quad a+b>0, \\ & \text{If } b<0, \text{ then } -\frac{c}{b}<\frac{c(a+2b)}{2b(a+b)}<-\frac{c}{a+b}, \\ & \text{If } b>0, \text{ then } -\frac{c}{a+b}<\frac{c(a+2b)}{2b(a+b)}<-\frac{c}{b}. \end{aligned} \tag{34}$$

Let us notice that functon $\bar{\pi}^P(C_P)$ is continuous function of C_P and equal to $\pi^P(C_P, W)$ when $C_P \in (0, \bar{C}_P)$. Further in this proof we consider function $\bar{\pi}^P(C_P)$ and focus on the behavior of this function in the interval $C_P \in (0, \bar{C}_P)$.

When $C_P \to -\frac{c}{b}$ function $\bar{\pi}^P(C_P) \to +\infty$ and when $C_P \to -\frac{c}{a+b}$ then $\bar{\pi}^P(C_P) \to -\infty$. The first and second order derivatives of function $\bar{\pi}^P(C_P)$ with respect to C_P can be found as follows:

$$\begin{aligned} \frac{\partial \bar{\pi}^P(C_P)}{\partial C_P} &= \frac{ac}{(c+bC_P)(c+aC_P+bC_P)}-d, \\ \frac{\partial^2 \bar{\pi}^P(C_P)}{\partial^2 C_P} &= -\frac{ac(ac+2bc+2b^2C_P+2abC_P)}{(c+bC_P)^2(c+(a+b)C_P)^2}. \end{aligned} \tag{35}$$

First we consider the case when $b<0$. Let us notice that in this case $-\frac{c}{a+b}<0$ and $\bar{C}_P < -\frac{c}{b}$. If $4b^2+4ab+acd \le 0$ then function $\bar{\pi}^P(C_P)$ is increasing in the interval $C_P \in (-\frac{c}{a+b}, -\frac{c}{b})$ and the optimal C_P for the primary user is equal to $\bar{C}_P$. Otherwise, since function $\bar{\pi}^P(C_P)$ is concave when $C_P \le \frac{c(a+2b)}{2b(a+b)}$ and convex when $C_P \ge \frac{c(a+2b)}{2b(a+b)}$, and the point $C_P = \frac{c(a+2b)}{2b(a+b)}$ belongs to the interval $(-\frac{c}{b}, -\frac{c}{a+b})$, there is one local maximum at the point $C_P = \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$ and one local minimum at the point $C_P = \frac{-\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$. We can conclude that function $\bar{\pi}^P(C_P)$ is strictly increasing in the interval $C_P \in (-\frac{c}{a+b}, \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}]$, strictly decreasing when $C_P \in [\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}, \frac{-\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}]$ and strictly increasing in the interval $C_P \in [\frac{-\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}, -\frac{c}{b})$. Thus, if local maximum $C_P = \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$ belongs to the interval $(0, \bar{C}_P)$ and the value of function $\bar{\pi}^P(C_P)$ at this point is greater than in the case when

$C_P = \bar{C}_P$, then the optimal price assigned by the primary user C_P^* is equal to $\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$.

If $b = 0$, then $\frac{\partial^2 \bar{\pi}^P(C_P)}{\partial^2 C_P} < 0$ and therefore the function $\bar{\pi}^P(C_P)$ is strictly concave in the interval $C_P \in (-\frac{c}{a+b}, +\infty)$. One can notice that $-\frac{c}{a+b} < 0$ and function $\bar{\pi}^P(C_P)$ is strictly increasing in the interval $C_P \in (-\frac{c}{a+b}, \frac{1}{d} - \frac{c}{a}]$ and strictly decreasing in the interval $C_P \in [\frac{1}{d} - \frac{c}{a}, +\infty)$. Thus, if $\frac{1}{d} - \frac{c}{a}$ belongs to the interval $(0, \bar{C}_P)$, then the optimal price assigned by the primary user $C_P^* = \frac{1}{d} - \frac{c}{a}$.

Finally, in the case $b > 0$ we can see that $-\frac{c}{b} < -\frac{c}{a+b} < 0$. Function $\bar{\pi}^P(C_P)$ is convex when $C_P \leq \frac{c(a+2b)}{2b(a+b)}$ and concave when $C_P \geq \frac{c(a+2b)}{2b(a+b)}$, and point $C_P = \frac{c(a+2b)}{2b(a+b)}$ belongs to the interval $C_P \in (-\frac{c}{b}, -\frac{c}{a+b})$. One can notice that $4b^2 + 4ab + acd > 0$ when $b > 0$, and therefore function $\bar{\pi}^P(C_P)$ has one local minimum $C_P = \frac{-\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$ in the interval $C_P \in (-\infty, -\frac{c}{b}$ and one local maximum $C_P = \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$ in the interval $C_P \in (-\frac{c}{a+b}, +\infty)$. In particular it means that function $\bar{\pi}^P(C_P)$ is strictly increasing when $C_P \in (-\frac{c}{a+b}), \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}]$ and strictly decreasing when $C_P \in (\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}], +\infty$. Thus, if local maximum $C_P = \frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$ belongs to the interval $(0, \bar{C}_P)$, then the optimal price assigned by the primary user C_P^* is equal to $\frac{\sqrt{acd(4b^2+4ab+acd)}-(a+2b)}{2bd(a+b)}$.

One can easily notice that in each case described above optimal price C_P^* assigned by the primary user is unique, i.e. there is no such $\hat{C}_P \in (0, \bar{C}_P)$ for which $\bar{\pi}^P(\hat{C}_P) \geq \bar{\pi}^P(C_P^*)$ if one of the conditions (29), (30) or (31) is satisfied. Conversely, if none of the conditions (29), (30) or (31) is fulfilled, then C_P^* is not optimal or not unique. In order to maximize own payoff, the i-th secondary user applies its best response strategy $\bar{P}_i^S(C_P)$ and therefore $P_i^{S*} = \bar{P}_i^S(C_P^*)$.

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